THE NORMAL DISTRIBUTION

Text

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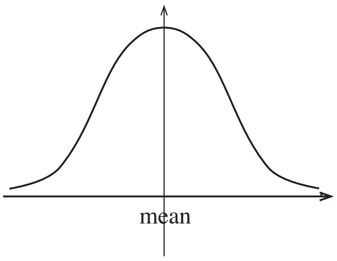
The Normal Distribution

1 Looking at your Data

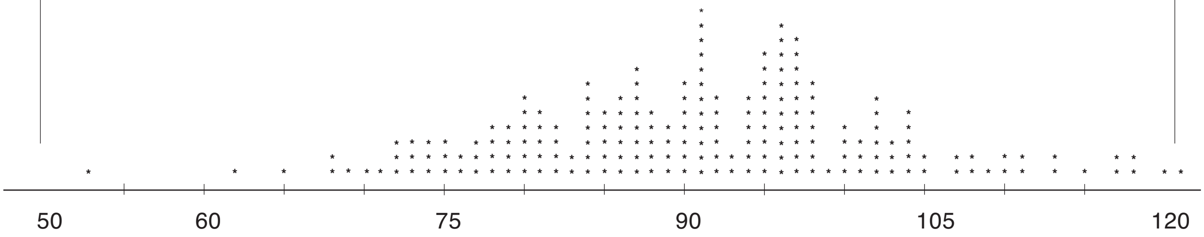


The **normal distribution** is a very useful way of summarising and working with data sets. Many data sets, such as IQ (intelligence quotient), shoe size, length of travel time between two locations, etc. can be approximated by a distribution of the form shown opposite, that is, bell-shaped and symmetric about the mean.

The following data gives the length from top to tail (in millimetres) of a large group of frogs. This has been run through a computer package so you can see some useful facts about the data.

Frog Data

83	69	97	53	89	95	105	80	76	117	74
91	100	77	110	68	118	87	97	78	100	95
73	103	96	72	71	99	121	81	104	68	89
87	96	87	72	79	102	98	97	88	87	86
103	79	104	105	91	82	102	75	95	90	62
65	97	86	97	111	98	92	74	88	84	80
95	96	92	95	100	90	91	95	75	70	84
80	98	96	94	101	85	113	96	103	98	95
84	84	97	95	108	94	79	81	92	85	87
90	85	82	81	97	79	90	90	94	98	73
91	91	107	102	89	85	98	84	91	90	86
113	86	93	77	100	96	90	97	109	102	84
85	87	97	92	107	102	104	94	93	75	96
91	117	91	87	118	96	89	88	111	120	92
76	94	104	80	77	94	84	78	73	92	81
83	104	91	91	96	88	115	96	74	88	86
80	98	101	95	96	102	78	97	80	87	82
72	78	108	91	91	91	110	86	101	81	97
82	97									



	N	MEAN	MEDIAN	TRMEAN	STDEV	MIN	MAX	Q1	Q3
Frogs	200	90.905	91.000	90.822	11.701	53.000	121.000	83.250	97.000

Stem and leaf of frogs (including cumulative frequency in both directions)

Leaf Unit = 1.0 N = 200

```
                      1 |   5 | 3
                      1 |   5 |
      ↓               2 |   6 | 2
 cumulative           6 |   6 | 5 8 8 9
  frequency          17 |   7 | 0 1 2 2 2 3 3 3 4 4 4
                     33 |   7 | 5 5 5 6 6 7 7 7 8 8 8 8 9 9 9 9
                     57 |   8 | 0 0 0 0 0 0 1 1 1 1 1 2 2 2 2 3 3 4 4 4  4 4 4 4
                     85 |   8 | 5 5 5 5 5 6 6 6 6 6 6 7 7 7 7 7 7 7 7 8 8 8 8 8 9 9 9 9
                   (34) |   9 | 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 2 2 2 2 2 2 3 3 4 4 4 4 4 4
                     81 |   9 | 5 5 5 5 5 5 5 5 5 6 6 6 6 6 6 6 6 6 6 6 7 7 7 7 7 7 7 7 7 7 7 7 8 8 8 8 8 8 8 9
                     41 |  10 | 0 0 0 0 1 1 1 2 2 2 2 2 2 3 3 3 4 4 4 4 4
 cumulative          20 |  10 | 5 5 7 7 8 8 9
  frequency          13 |  11 | 0 0 1 1 3 3
      ↑               7 |  11 | 5 7 7 8 8
                      2 |  12 | 0 1
```

In the computer analysis you will see that most of the frogs are close to the mean value, with fewer at the extremes. This 'bell-shaped' pattern of distribution is typical of data which follows a **normal distribution**. To obtain a perfectly shaped and symmetrical distribution you would need to measure thousands of frogs.

You may notice that median $\approx$ mean $\approx$ mode, as might be expected for a symmetrical distribution. From the analysis of data you also see that the mean is 90.9 mm and the standard deviation is 11.7 mm. Now look at how much of the data is close to the mean, i.e. within one standard deviation (SD or s.d.) of it. From the stem and leaf table you can see that 74 frogs have a length within one standard deviation above the mean and 59 within a SD below the mean.

Altogether, 133 frogs are + or − one SD from the mean, which is 66.5% of the population sample.

IQ test scores, and the results of many other standard tests, are designed to be normally distributed with mean 100 and standard deviation 15.

In fact, the normal distribution is designed so that 68% of the population lie within 1 standard deviation of the mean, 95% within 2 standard deviations, and can be summarised as:

Deviation from the mean	0-1 SD	1-2 SD	2-3 SD	over 3
Proportion of population	34%	$13\frac{1}{2}$%	$2\frac{1}{2}$%	negligible

Therefore statements such as the following can be made:

> '68% of all people should achieve an IQ score between 85 and 115.'
>
> 'Only $2\frac{1}{2}$% of people should have an IQ score less than 70.'
>
> 'Only 1 in a 1000 people have an IQ greater than 145.'

Exercise

1. A survey showed that the average height of 16-19 year olds was approximately 169 cm with SD 9 cm. Assuming the data follows a normal distribution, find:

   (a) the percentage of sixth formers taller than 187 cm;

   (b) the percentage of sixth formers smaller than 160 cm;

   (c) in a sixth form of 300, the number of students smaller than 151 cm.

   (Note these are not truly normal, as the pattern for girls and boys is different.)

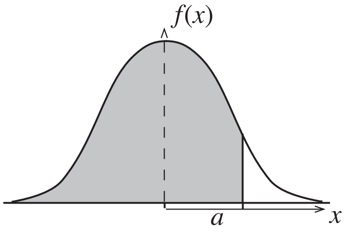
2 The P.D.F. of the Normal

If you could work in only whole numbers of SDs, the number of problems that could be solved would be limited. To calculate the proportions or probabilities of lying within so many SDs of the mean, you need to know what is called the **probability density function**, p.d.f.

A distribution for a continuous populations is defined by its p.d.f., denoted by $f(x)$ and is such that

$$P(X \le a) = \int_{-\infty}^{a} f(x)\,dx$$

and
$$\int_{-\infty}^{\infty} f(x) = 1$$



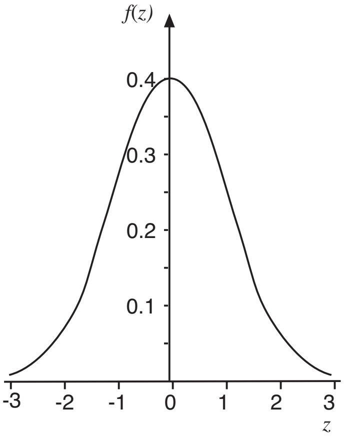
Although the normal distribution is symmetric, not all distributions are.

The famous German mathematician, *Carl Friedrich Gauss* (1777-1855) developed the normal distribution, sometimes called the **Gaussian distribution**.

It is given by the formula

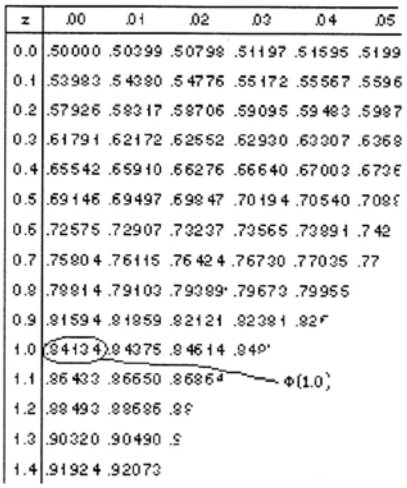
$$f(z) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}z^2}$$

z is called the **standard normal variate** and represents a normal distribution with mean 0 and SD 1. The graph of the function is shown opposite.



Note that the function $f(z)$ has no value for which it is zero, i.e. it is possible, though very unlikely, to have very large or very small values occurring.

In order to find the probabilities of all possible SDs from the mean you would have to integrate the function between the values, that is, find the area under the graph between certain limits. This is a tedious task and to avoid this, tables of the function are commonly used and these are readily available on some calculators.



z	.00	.01	.02	.03	.04	.05
0.0	.50000	.50399	.50798	.51197	.51595	.5199
0.1	.53983	.54380	.54776	.55172	.55567	.5596
0.2	.57926	.58317	.58706	.59095	.59483	.5987
0.3	.61791	.62172	.62552	.62930	.63307	.6368
0.4	.65542	.65910	.66276	.66640	.67003	.673
0.5	.69146	.69497	.69847	.70194	.70540	.708
0.6	.72575	.72907	.73237	.73565	.73891	.742
0.7	.75804	.76115	.76424	.76730	.77035	.77
0.8	.78814	.79103	.79389	.79673	.79955	
0.9	.81594	.81859	.82121	.82381	.82	
1.0	.84134	.84375	.84614	.849		
1.1	.86433	.86650	.86864			
1.2	.88493	.88686	.8			
1.3	.90320	.90490	.			
1.4	.91924	.92073				

$\Phi(1.0)$

We define

$$\Phi(z) = P(Z < z)$$

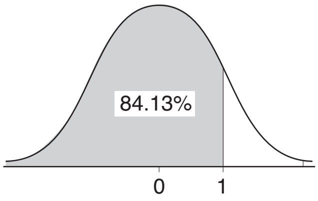
$$= \int_{-\infty}^{z} f(z)\, dz$$

So $\Phi(z)$ denotes the cumulative probability up to the value of z.

For positive z, the function gives you the probability of being less than z SDs above the mean.

For example, $\Phi(1.0) = 0.84313$, therefore 84.13% of the distribution is less than one SD above the mean.

Tables usually give the area to the left of z and only for values above zero. This is because symmetry enables you to calculate all other values.



Worked Example 1

What is the probability of being less than 1.5 SDs below the mean i.e. $\Phi(-1.5)$?

Solution

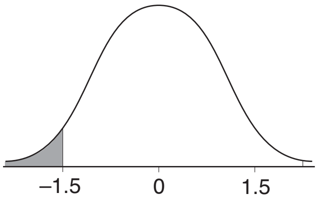
From tables,

$$\Phi(+1.5) = 0.93319$$

and by symmetry,

$$\Phi(-1.5) = 1 - 0.93319 = 0.06681$$

i.e. about 6.7%.



A random variable, Z, which has this p.d.f. is denoted by

$$\boxed{Z \sim N(0,1)}$$

showing that it is a normal distribution with mean 0 and standard deviation 1.

This is often referred to as the **standardised** normal distribution.

Worked Example 2

If $Z \sim N(0,1)$, find

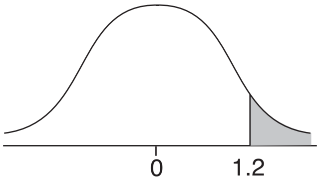
(a) $P(Z > 1.2)$ (b) $P(-2.0 < Z < 2.0)$ (c) $P(-1.2 < Z < 1.0)$

Solution

(a) $P(Z > 1.2) = 1 - \Phi(1.2)$

$= 1 - 0.88493$ (from tables)

$= 0.11507$



(b) $P(-2.0 < Z < 2.0)$

$= P(Z < 2.0) - P(Z < -2.0)$

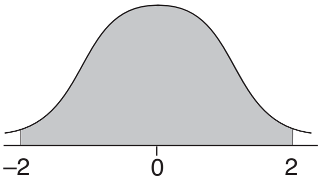
$= \Phi(2.0) - P(Z > 2.0)$

$= \Phi(2.0) - (1 - P(Z < 2.0))$

$= 2\Phi(2.0) - 1$

$= 2 \times 0.97725 - 1$

$= 0.9545$



(c) $P(-1.2 < Z < 1.0)$

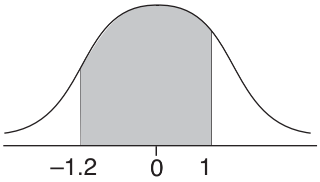
$= P(Z < 1.0) - P(Z < -1.2)$

$= P(Z < 1.0) - P(Z > 1.2)$

$= \Phi(1.0) - (1 - \Phi(1.2))$

$= 0.84134 - (1 - 0.88493)$

$= 0.72627$



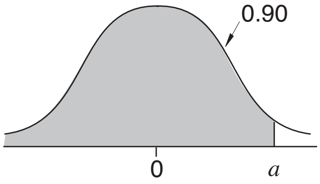
You can also use the tables to find the value of a when $P(Z > a)$ is a given value and $Z \sim N(0,1)$. This is illustrated in the next worked example.

Worked Example 3

If $Z \sim N(0,1)$, find a such that

(a) $P(Z < a) = 0.90$

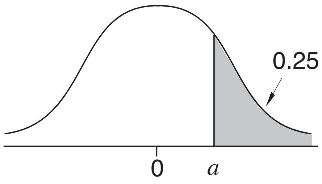
(b) $P(Z > a) = 0.25$



Solution

(a) Here $\Phi(a) = 0.90$, and from the tables

$a \approx 1.28$



(b) Here $\Phi(a) = 1 - 0.25 = 0.75$ and from the tables
$a \approx 0.67$

Note

If you have not used the Normal distribution before, you will need to practise finding probabilities such as those given in the Exercises below. It is *always* very helpful to draw a sketch and indicate on it the area to be found. This will enable you to estimate the probability and ensure that there is no misconception when you evaluate the probability from the tables.

Exercise

1. If $Z \sim N(0,1)$, find

(a) $P(Z > 0.82)$ (b) $P(Z < 0.82)$ (c) $P(Z > -0.82)$

(d) $P(Z < -0.82)$ (e) $P(-0.82 < Z < 0.82)$ (f) $P(-1 < Z < 1)$

(g) $P(-1 < Z < 1.5)$ (h) $P(0 < Z < 2.5)$ (i) $P(Z < -1.96)$

(j) $P(-1.96 < Z < 1.96)$

3 Transformation of Normal P.D.Fs

The method needed to transform any normal variable to the standardised variable is illustrated in the example below.

Worked Example 1

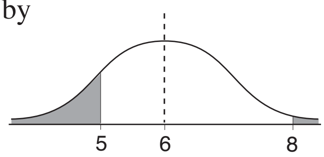
Eggs laid by a particular hen are known to have lengths normally distributed, with mean 6 cm and standard deviation 1.4 cm. What is the probability of:

(a) finding an egg bigger than 8 cm in length;

(b) finding an egg smaller than 5 cm in length?

Solution

(a) The number of SDs that 8 is above the mean is given by



$$z = \frac{x - \mu}{\sigma} = \frac{8 - 6}{1.4} = 1.429$$

but $\Phi(1.43) = 0.92364$ (from tables)

so $P(X > 8) = 1 - 0.92364 = 0.07646$.

(b) $$z = \frac{5 - 6}{1.4} = -0.7143$$

but $\Phi(0.7143) = 0.7625$ (from tables using interpolation),

so $P(X < 5) = 1 - 0.7625 = 0.2375$

Note that in order to find the probability you need to establish whether you need the area greater than a half or less than a half. Drawing a diagram will help.

When a variable X follows a **normal distribution**, with mean m and variance σ^2, this is denoted by

$$\boxed{X \sim N\left(\mu, \sigma^2\right)}$$

So in the last example, $X \sim N\left(6, 1.4^2\right)$.

To use normal tables, the transformation

$$\boxed{Z = \frac{X - \mu}{\sigma}}$$

is used. This ensures that Z has mean 0 and standard deviation 1, and the tables are then valid.

The tallest person in recorded history for whom there is irrefutable evidence of height is Robert Wadlow (1918 - 1940), an American who was born and raised in Illinois. He reached the height of 2.72 m (8 ft 11.1 in)

Using the UK data on heights in the Exercise in section 6.1, the z value for Robert Wadlow's height is

$$z = \frac{272 - 167.3}{9.1} \approx 11.5$$

So his height is 11.5 SDs above the mean. The most accurate tables show that 6 SDs is only exceeded with a probability of 10^{-10}, so it is extremely unlikely that a taller person will ever appear in our lifetime!

Worked Example 2

If $X \sim N(4,9)$, find

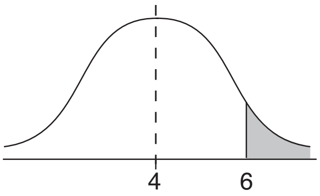
(a) $P(X > 6)$ (b) $P(X > 1)$

Solution

Now $Z = \dfrac{X-\mu}{\sigma} = \dfrac{X-4}{3}$.

(a) Hence

$$\begin{aligned} P(X>6) &= 1 - P(X<6) \\ &= 1 - \Phi\left(\frac{6-4}{3}\right) \\ &= 1 - \Phi(0.67) \\ &= 1 - 0.74857 \\ &= 0.25143 \end{aligned}$$

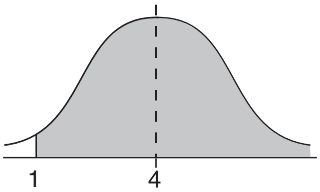


(b) $P(X > 1) = P(X < 7)$ (by symmetry)

$$= \Phi\left(\frac{7-4}{3}\right)$$

$$= \Phi(1)$$

$$= 0.84134$$



Exercises

1. If $X \sim N(200, 625)$, find
   (a) $P(X > 250)$ (b) $P(175 < X < 225)$ (c) $P(X < 275)$

2. If $X \sim N(6, 4)$, find
   (a) $P(X > 8)$ (b) $P(4 < X < 8)$ (c) $P(5 < X < 9)$

3. If $X \sim (-10, 36)$, find
   (a) $P(X < 0)$ (b) $P(-12 < X < -8)$ (c) $P(-15 < X < 0)$

4. Components in a personal stereo are normally distributed with a mean life of 2400 hours with SD 300 hours. It is estimated that the average user listens for about 1000 hours in one year. What is the probability that a component lasts for more than three years.

5. The maximum flow of a river in Africa during the 'rainy season' was recorded over a number of years and found to be distributed

   $$N\left(6300,\ 1900^2\right)\ \mathrm{m^3\,s^{-1}}$$

   For the banks to burst a flow of $8700\ \mathrm{m^3\,s^{-1}}$ is required. What is the probability of this happening in a particular year?

6. IQs are designed to be $N(100, 225)$. To join Mensa an IQ of 138 is required. What percentage of the population are eligible to join?

   A psychologist claims that any child with an IQ of $150+$ is 'gifted'. How many 'gifted' children would you expect to find in a school of 1800 pupils?

7. Rainfall in a particular area has been found to be $N\left(850,\ 100^2\right)$ mm over the years. What is the probability of rainfall exceeding 1000 mm?

8. In a verbal reasoning test on different ethnic groups, one group was found to have scores distributed $N\left(98.42,\ 15.31^2\right)$. Those with a score less than 80 were deemed to be in need of help. What percentage of the overall group were in need of help?

4

More Complicated Examples

The following examples illustrate some of the many uses and applications of the normal distribution.

Worked Example 1

A machine produces bolts which are $N(4,\ 0.09)$, where measurements are in mm. Bolts are measured accurately and any which are smaller than 3.5 mm or bigger than 4.4 mm are rejected. Out of a batch of 500 bolts how many would be acceptable?

Solution

$$P(X < 4.4) = \Phi\left[\frac{(4.4 - 4)}{0.3}\right] \approx \Phi(1.33) = 0.90824$$

$$P(X < 3.5) = \Phi\left[\frac{(3.5 - 4)}{0.3}\right] \approx \Phi(-1.67) = 0.04746$$

Hence $\quad P(3.5 < X < 4.4) \approx 0.90824 - 0.04746$

$$= 0.86078$$

The number of acceptable items is therefore

$0.86078 \times 500 = 430.39 \approx 430$ (to nearest whole number).)

Worked Example 2

IQ tests are measured on a scale which is $N(100,\ 225)$. A woman wants to form an 'Eggheads Society' which only admits people with the top 1% of IQ scores. What would she have to set as the cut-off point in the test to allow this to happen?

Solution

From tables you need to find z such that $\Phi(z) = 0.99$.

This is most easily carried out using a 'percentage points of the normal distribution' table, which gives the values directly.

Now $\quad \Phi^{-1}(0.99) = 2.3263$

which is an alternative way of saying that

$$\Phi(2.3263) = 0.99$$

(Check this using the usual tables.)

This means that

$$\frac{x - 100}{\sqrt{225}} = 2.3263$$

Hence $\quad x = 100 + 2.3263 \times 15$

$$= 134.8945 \approx 134.9$$

Worked Example 3

A manufacturer does not know the mean and SD of the diameters of ball bearings he is producing. However, a sieving system rejects all bearings larger than 2.4 cm and those under 1.8 cm in diameter. Out of 1000 ball bearings 8% are rejected as too small and 5.5% as too big. What is the mean and standard deviation of the ball bearings produced?

Solution

Assume a normal distribution of

$$\Phi^{-1}(1-0.08) = 1.4$$

so 1.8 is 1.4 standard deviations below mean.

Also $\Phi^{-1}(1-0.055) = 1.6$

so 2.4 is 1.6 standard deviations above the mean.

This can be written as two simultaneous equations and solved:

$$\mu + 1.6\sigma = 2.4$$

$$\mu - 1.4\sigma = 1.8$$

Subtracting,

$$3.0\sigma = 0.6$$

$$\Rightarrow \quad \sigma = 0.2$$

Using the first equation,

$$\mu + (1.6 \times 0.2) = 2.4$$

$$\Rightarrow \quad \mu = 2.4 - (1.6 \times 0.2)$$

$$\Rightarrow \quad \mu = 2.08$$

So diameters are distributed $N\left(2.08,\ 0.2^2\right)$.

Exercises

1. Bags of sugar are sold as 1 kg. To ensure bags are not sold underweight the machine is set to put a mean weight of 1004 g in each bag. The manufacturer claims that the process works to a standard deviation of 2.4. What proportion of bags are underweight?

2. Parts for a machine are acceptable within the 'tolerance' limits of 20.5 to 20.6 mm. From previous tests it is known that the machine produces parts to $N\left(20.56,\ (0.02)^2\right)$.
   Out of a batch of 1000 parts how many would be expected to be rejected?

3. Buoyancy aids in watersports are tested by adding increasing weights until they sink. A club has two sets of buoyancy aids. One set is two years old, and should support weights according to $N(6.0,\ 0.64)$ kg; the other set is five years old and

should support weights of $N(4.5,\ 1.0)$ kg. All the aids are tested and any which are unable to support at least 5 kg are thrown out.

(a) If there are 24 two-year-old aids, how many are still usable?

(b) If there are 32 five-year-old aids how many are still usable?

4. Sacks of potatoes are packed by an automatic loader with mean weight 114 lb. In a test it was found that 10% of bags were over 116 lb. Use this to find the SD of the process. If the machine is now adjusted to a mean weight of 113 lb, what % are now over 116 lb if the SD remains unaltered?

5. In a soap making process it was found that $6\frac{2}{3}\%$ of bars produced weighed less than 90.50 g and 4% weighed more than 100.25 g.

   (a) Find the mean and the SD of the process.

   (b) What % of the bars would you expect to weigh less than 88 g?

6. A light bulb manufacturer finds that 5% of his bulbs last more than 500 hours. An improvement in the process meant that the mean lifetime was increased by 50 hours . In a new test, 20% of bulbs now lasted longer than 500 hours.

   Find the mean and standard deviation of the original process.

7. The masses of plums from a certain orchard have mean 24g and standard deviation 5g. The plums are graded small, medium or large. All plums over 28g in mass are regarded as large and the rest equally divided between small and medium.

   Assuming a normal distribution find:

   (a) the proportion of plums graded large;

   (b) the upper limit of the masses of the plums in the small grade.

8. The weights of pieces of home made fudge are normally distributed with mean 34 g and standard deviation 5 g.

   (a) What is the probability that a piece selected at random weighs more than 40g?

   (b) For some purposes it is necessary to grade the pieces as small, medium or large. It is decided to grade all pieces weighing over 40 g as large and to grade the heavier half of the remainder as medium. The rest will be graded as small. What is the upper limit of the small grade?

9. Yuk Ping belongs to an athletics club. In javelin throwing competitions her throws are normally distributed with mean 41.0 m and standard deviation 2.0 m.

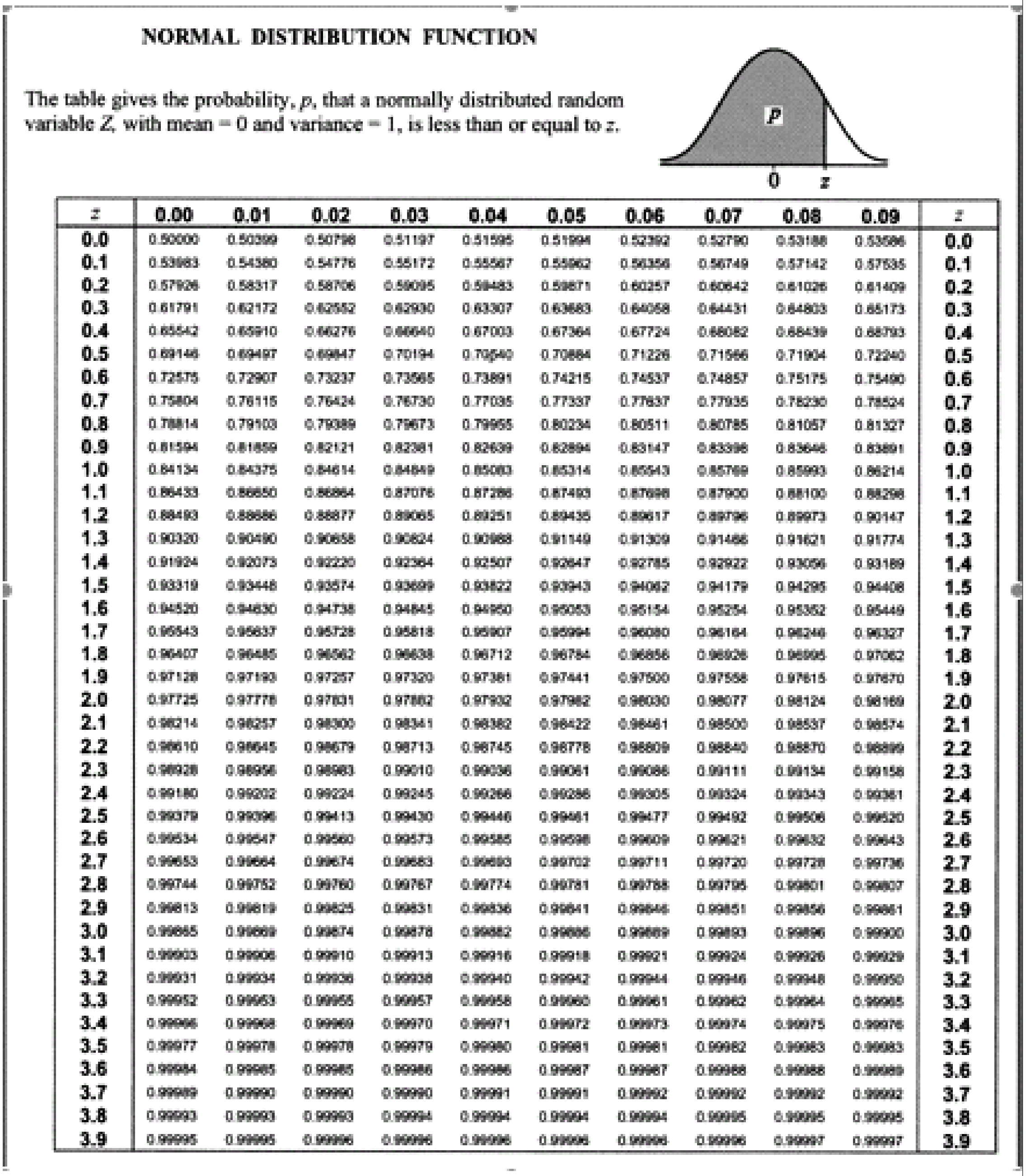
   (a) What is the probability of her throwing between 40 m and 46 m?

   (b) What distance will be exceeded by 60% of her throws?

   Gwen belongs to the same club. In competitions 85% of her javelin throws exceed 35 m and 70% exceed 37.5 m. Her throws are normally distributed.

   (c) Find the mean and standard deviation of Gwen's throws, each correct to two significant figures.

   (d) The club has to choose one of these two athletes to enter a major competition. In order to qualify for the final round it is necessary to achieve a throw of at least 48 m in the preliminary rounds. Which athlete should be chosen and why?



NORMAL DISTRIBUTION FUNCTION

The table gives the probability, *p*, that a normally distributed random variable *Z*, with mean = 0 and variance = 1, is less than or equal to *z*.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	*z*
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586	**0.0**
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535	**0.1**
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409	**0.2**
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173	**0.3**
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793	**0.4**
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240	**0.5**
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490	**0.6**
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524	**0.7**
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327	**0.8**
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891	**0.9**
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214	**1.0**
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298	**1.1**
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147	**1.2**
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774	**1.3**
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189	**1.4**
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408	**1.5**
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449	**1.6**
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327	**1.7**
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062	**1.8**
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670	**1.9**
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169	**2.0**
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574	**2.1**
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899	**2.2**
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158	**2.3**
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361	**2.4**
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520	**2.5**
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643	**2.6**
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736	**2.7**
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807	**2.8**
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861	**2.9**
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900	**3.0**
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929	**3.1**
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950	**3.2**
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965	**3.3**
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976	**3.4**
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983	**3.5**
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989	**3.6**
3.7	0.99989	0.99990	0.99990	0.99990	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992	**3.7**
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995	**3.8**
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997	**3.9**