SINE RULE AND COSINE RULE

Text

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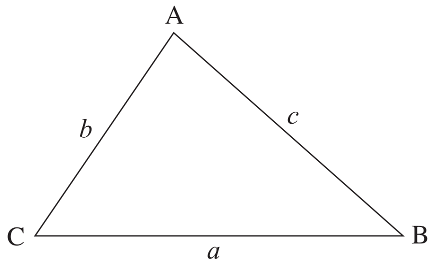
Sine Rule and Cosine Rule

1 Sine and Cosine Rules

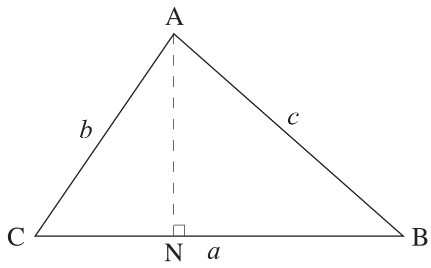
In the triangle ABC, the side opposite angle A has length a, the side opposite angle B has length b and the side opposite angle C has length c.

The *sine rule* states

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



Proof of Sine Rule



If you construct the perpendicular from vertex A to meet side CB at N, then

$$\begin{aligned} AN &= c\sin B \quad (\text{from } \Delta\ ABN) \\ &= b\sin C \quad (\text{from } \Delta\ ACN) \end{aligned}$$

Hence

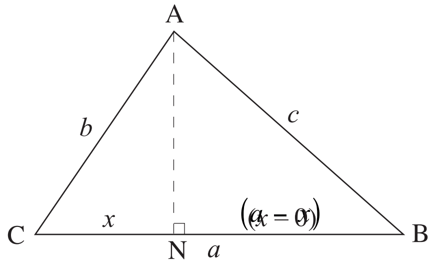
$$c\sin B = b\sin C \Rightarrow \frac{\sin B}{b} = \frac{\sin C}{c}$$

similarly for $\frac{\sin A}{a}$.

The *cosine rule* states

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ b^2 &= c^2 + a^2 - 2ca\cos B \\ c^2 &= a^2 + b^2 - 2ab\cos C \end{aligned}$$

Proof of Cosine Rule



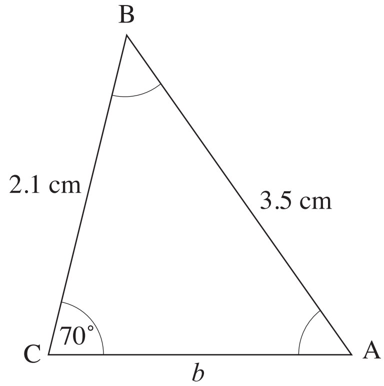
If CN $= x$, then NB $= a - x$ and

$$\begin{aligned} c^2 &= AN^2 + (a - x)^2 \quad \text{when } x = CN \\ &= (b\sin C)^2 + (a - b\cos C)^2, \quad \text{since } x = b\cos C \\ &= b^2\sin^2 C + b^2\cos^2 C - 2ab\cos C + a^2 \\ &= b^2(\sin^2 C + \cos^2 C) + a^2 - 2ab\cos C \end{aligned}$$

i.e. $c^2 = b^2 + a^2 - 2ab\cos C$, since $\sin^2 C + \cos^2 C = 1$

Worked Example 1

Find the unknown angles and side length of the triangle shown.



Solution

Using the sine rule,

$$\frac{\sin A}{2.1} = \frac{\sin 70°}{3.5} = \frac{\sin B}{b}$$

From the first equality,

$$\sin A = \frac{2.1 \times \sin 70°}{3.5} = 0.5638$$

$$A = 34.32°$$

Since angles in a triangle add up to $180°$,

$$B = 180° - 70° - A = 75.68°$$

From the sine rule,

$$\frac{\sin 70°}{3.5} = \frac{\sin B}{b}$$

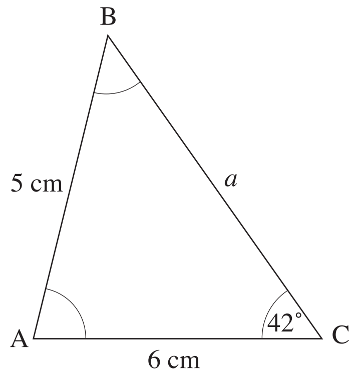
$$b = \frac{3.5 \times \sin B}{\sin 70°}$$

$$= \frac{3.5 \times \sin 75.68°}{\sin 70°}$$

$$= 3.61 \text{ cm}$$

Worked Example 2

Find two solutions for the unknown angles and side of the triangle shown.



Solution

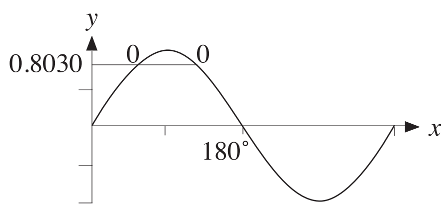
Using the sine rule,

$$\frac{\sin A}{a} = \frac{\sin B}{6} = \frac{\sin 42°}{5}$$

From the second equality,

$$\sin B = \frac{6 \times \sin 42°}{5} = 0.8030$$

A graph of $\sin x$ shows that between $0°$ and $180°$ there are two solutions for B.



These solutions are $B = 53.41°$ and, by symmetry, $B = 180 - 53.41$

$$= 126.59°$$

Solving for angle A we have

$$A = 180° - 42° - B$$

$$\text{when } B = 53.41°, \quad A = 84.59°$$

$$\text{when } B = 126.59°, \quad A = 11.41°$$

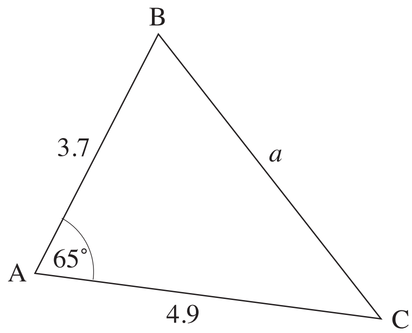
From the sine rule,

$$a = \frac{6 \times \sin A}{\sin B}$$

For $A = 84.59°$, $B = 53.41°$, $a = 7.44$ cmFor
$A = 11.41°$, $B = 126.59°$, $a = 1.48$ cm

Worked Example 3

Find the unknown side and angles of the triangle shown.



Solution

To find a, use the cosine rule:

$$a^2 = 4.9^2 + 3.7^2 - 2 \times 4.9 \times 3.7 \times \cos 65°$$

$$a^2 = 22.3759$$

$$a = 4.73 \quad \text{(to 2 d.p.)}$$

To find the angles, use the sine rule:

$$\frac{\sin 65°}{a} = \frac{\sin B}{4.9} = \frac{\sin C}{3.7}$$

$$\sin B = \frac{4.9 \times \sin 65°}{a} = \frac{4.9 \times \sin 65°}{4.73} = 0.9389$$

$$B = 69.86°$$

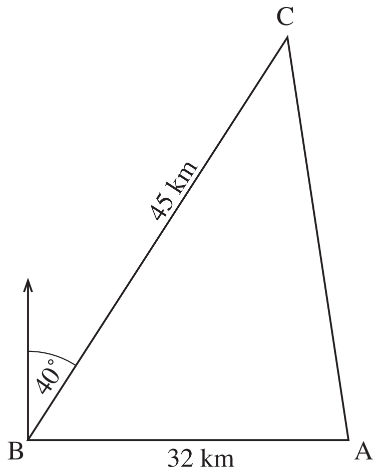
$$\sin C = \frac{3.7 \times \sin 65°}{a} = \frac{3.7 \times \sin 65°}{4.73} = 0.7090$$

$$C = 45.15° \quad \text{(alternatively, use } A + B + C = 180° \text{ to find C)}$$

Checking, $A + B + C = 65° + 69.86° + 45.15° = 180.01°$. The three angles should add to 180°; the extra .001° is due to rounding errors.

Worked Example 4

The diagram below, **not drawn to scale**, shows the journey of a ship which sailed from Port A to Port B and then to Port C. Port B is located 32 km due West of Port A and Port C is 45 km from Port B on a bearing of $040°$.



Calculate, giving your answers correct to 3 significant figures

(a) the distance AC

(b) the bearing of Port C from Port A.

Solution

(a) Using the COSINE rule,

$$AC^2 = BC^2 + BA^2 - 2 \times BC \times BA \times \cos C\hat{B}A$$

As angle CBA $= 90° - 40°$, then

$$AC^2 = 32^2 + 45^2 - 2 \times 32 \times 45 \times \cos 50°$$

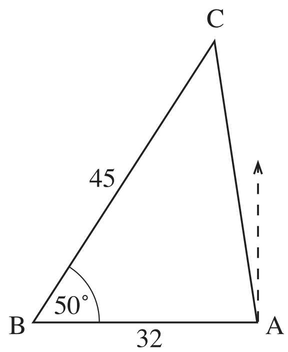
$$\approx 1197.77$$

$$AC \approx 34.6088$$

$$= 34.6 \text{ to 3 significant figures}$$

(b) The bearing of C from A is

$$270° + \text{angle BAC}$$



Using the SINE rule, $\dfrac{\sin BAC}{45} = \dfrac{\sin 50°}{34.6088}$

$$\sin BAC = \frac{45 \times \sin 50°}{34.6088}$$

$$\approx 0.9960$$

So angle BAC $\approx 84.903°$

$= 84.9°$ to 3 significant figures

Hence the bearing of Port A from Port B is

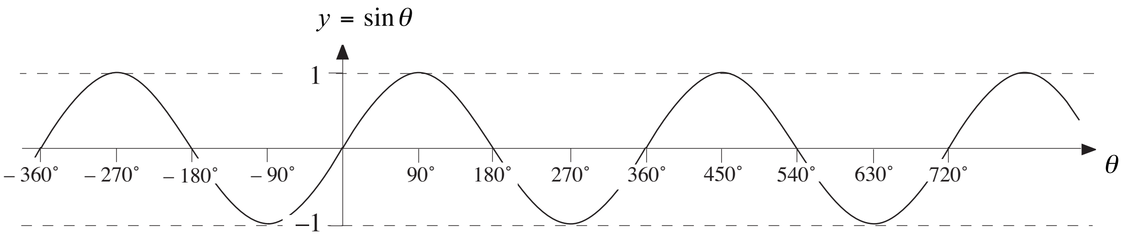
$270° + 84.9° = 354.9°$

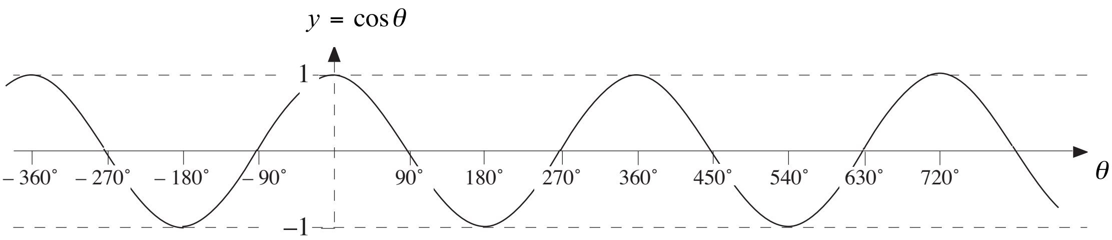
$= 355°$ to 3 significant figures

Some important values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ are shown in this table.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
$0°$	0	1	0
$30°$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$45°$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$60°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$90°$	1	0	infinite

The graphs of $\sin\theta$ and $\cos\theta$ for any angle are shown in the following diagrams.





The graphs are examples of *periodic functions.* Each basic pattern repeats itself every $360°$. We say that the *period* is $360°$.

Worked Example 5

An oil tanker leaves Town X, and travels on a bearing of $050°$ to Town Z, 50 km away. The tanker then travels to Town Y, 70 km away, on a bearing of $120°$.

(a) Draw a carefully labelled diagram of the tanker's journey, clearly showing the North line.

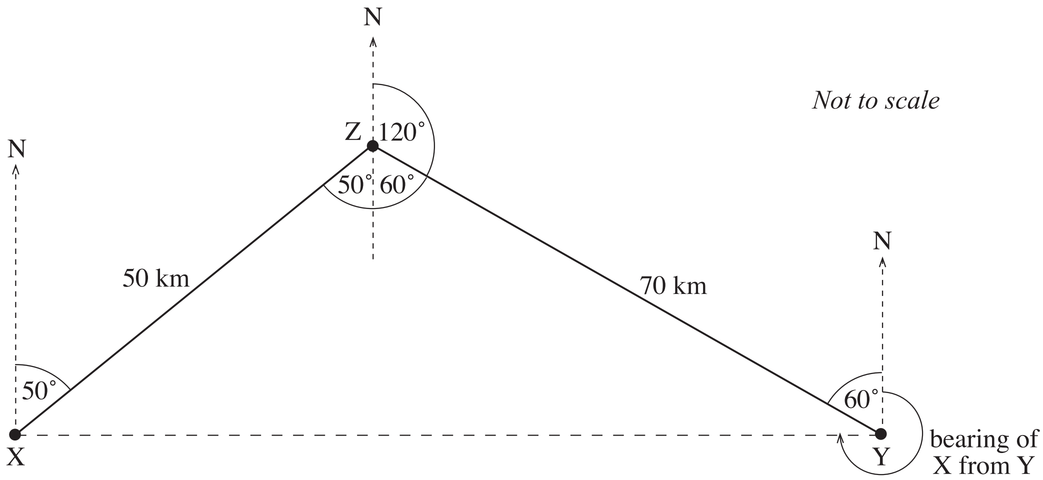
(b) Calculate the distance of Y from X, giving your answer to 3 significant figures.

(c) (i) On your diagram, <u>mark</u> the angle that shows the bearing of X from Y.

(ii) Calculate the bearing of X from Y, giving your answer to the nearest degree.

Solution

(a)



(b) In triangle XZY, angle XZY $= 50° + 60° = 110°$
so, using the cosine rule,

$$XY^2 = 50^2 + 70^2 - 2 \times 50 \times 70 \times \cos 110°$$
$$= 2500 + 4900 - 7000 \cos 110°$$
$$\approx 7400 - 7000 \times (-0.3420)$$
$$\approx 9794.14$$

$$XY \approx 98.965$$
$$= 99.0 \text{ to 3 significant figures.}$$

(c) (i) As marked on diagram.

(ii) We need to find the angle ZYX to determine the bearing of X from Y.

In triangle ZYX, using the sine rule,

$$\frac{\sin ZYX}{50} \approx \frac{\sin 110°}{99.0}$$

$$\sin ZYX \approx 0.4748$$

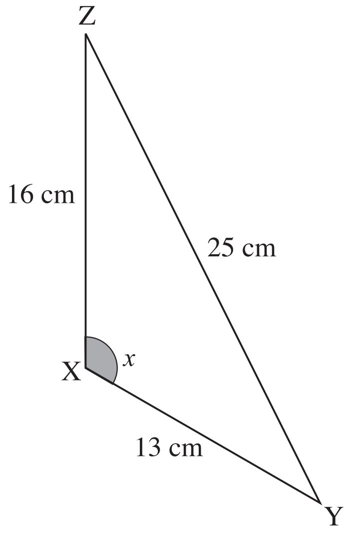
$$\text{angle } ZYX \approx 28.34° = 28° \text{ to the nearest degree}$$

Hence the bearing of X from Y

$$= 360° - (60° + 28°)$$
$$= 272°$$

Worked Example 6

Find the shaded angle in the triangle shown.



Solution

Using the cosine rule,

$$25^2 = 16^2 + 13^2 - 2 \times 16 \times 13 \cos x$$

Rearranging,

$$2 \times 16 \times 13 \cos x = 16^2 + 13^2 - 25^2$$

$$416 \cos x = -200$$

$$\cos x = \frac{-25}{52}$$

$$x \approx \cos^{-1}\left(\frac{-25}{52}\right)^\circ$$

$$= 118.7^\circ \quad \text{to 1 decimal place}$$

using the (SHIFT) and (COS) buttons on a calculator.

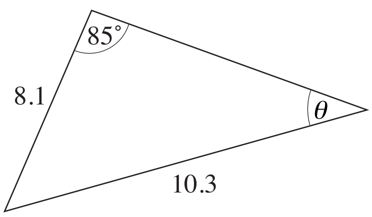
Note

As $\cos x$ is negative, the angle will be obtuse $(90^\circ < x < 180^\circ)$.

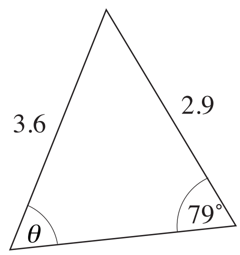
Exercises

1. For each of the triangles, find the unknown angle marked θ.

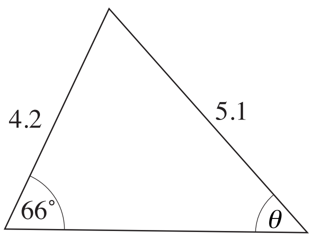
(a)



(b)



(c)



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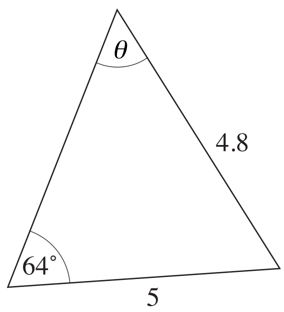
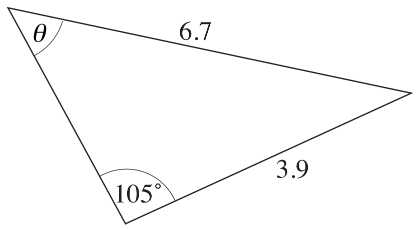
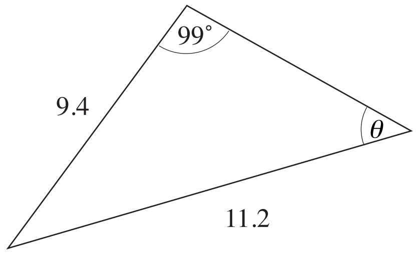


(e)

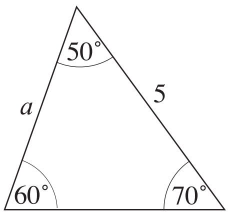


(f)



2. For each triangle, find the unknown side marked *a, b* or *c.*

(a)



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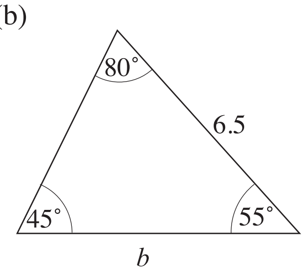

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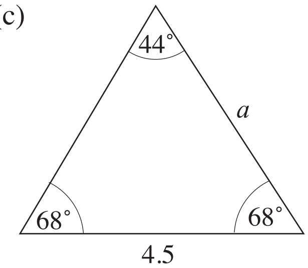

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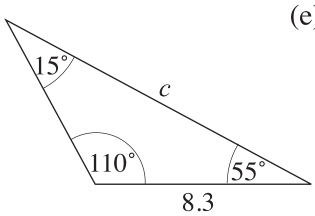

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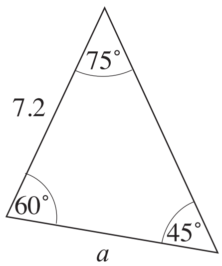

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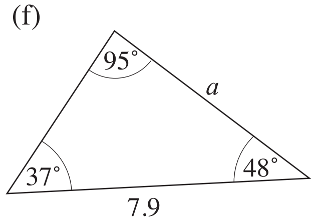
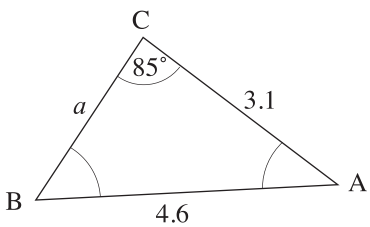


3. For each of the triangles, find the unknown angles and sides.

(a)



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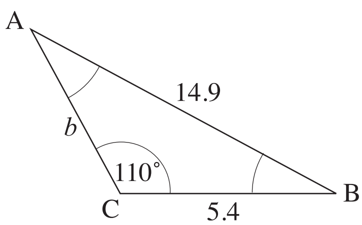

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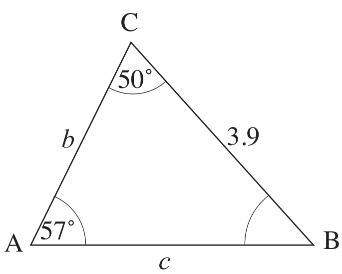

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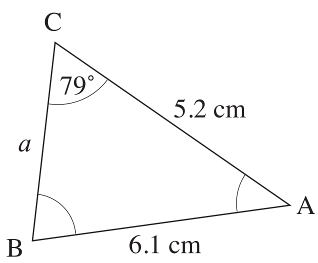
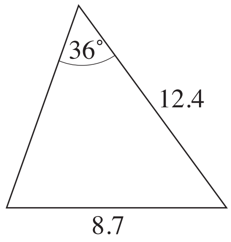


4. Which of the following triangles could have *two* solutions?

(a)



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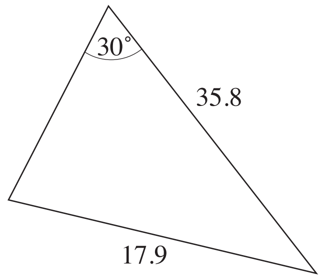

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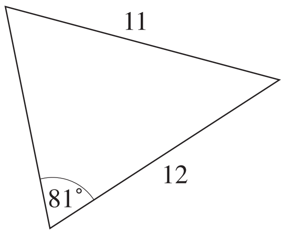

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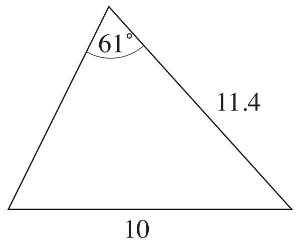
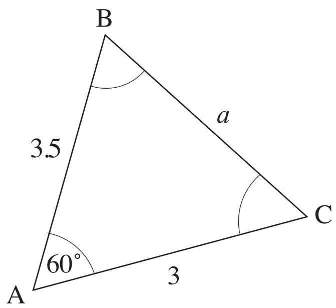


5. Find the remaining angles and sides of the triangle ABC if $A = 67°$, $a = 125$ and $c = 100$.

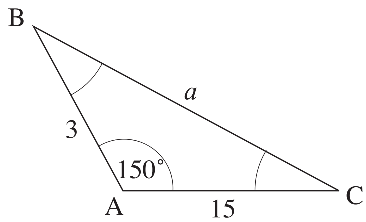
6. Find the remaining angles and sides of the triangle ABC if $B = 81°$, $b = 12$ and $c = 11$.

7. For each of the following triangles, find the unknown angles and sides.

(a)



(b)



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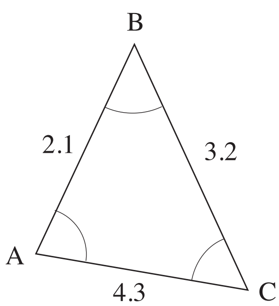

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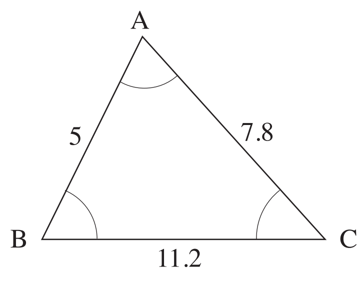
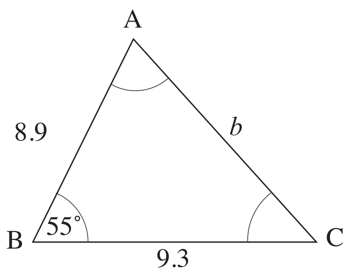


(e)



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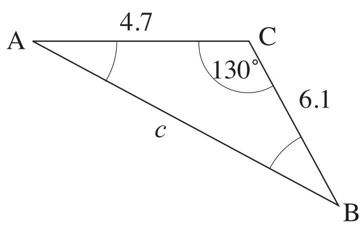
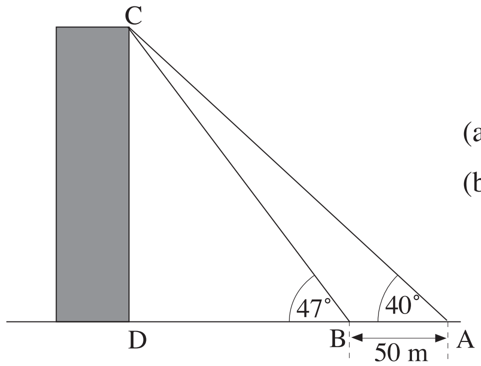


8. To calculate the height of a church tower, a surveyor measures the angle of elevation of the top of the tower from two points 50 metres apart. The angles are shown in the diagram.

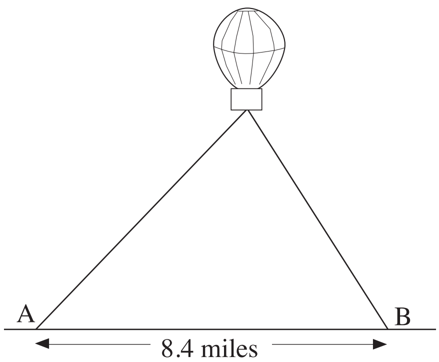


(a) Calculate the distance BC.

(b) Hence calculate the height of the tower CD.

9. The angles of elevation of a hot air balloon from two points, A and B, on level ground, are $24.2°$ and $46.8°$, respectively.

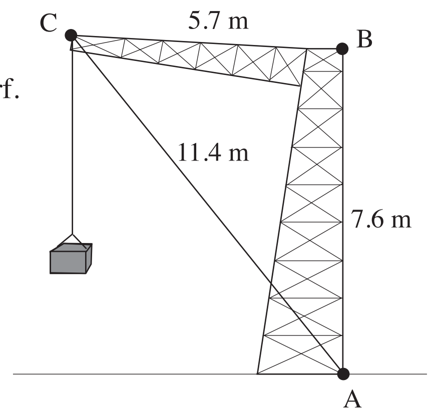
The points A and B are 8.4 miles apart, and the balloon is between the points in the same vertical plane. Find the height of the balloon above the ground.



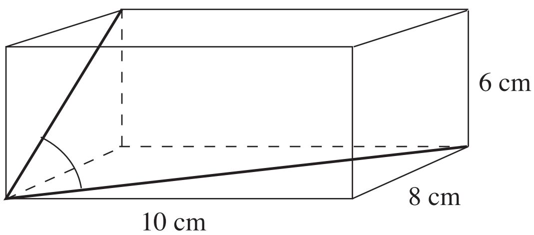
10. The diagram shows a crane working on a wharf. AB is vertical.

(a) Find the size of angle ABC.

(b) Find the height of point C above the wharf.

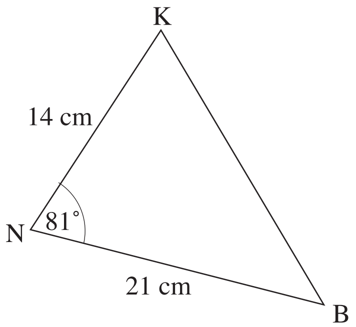


11. The rectangular box shown in the diagram has dimensions 10 cm by 8 cm by 6 cm. Find the angle θ formed by a diagonal of the base and a diagonal of the 8 cm by 6 cm side.



12. (a) Calculate the length KB.

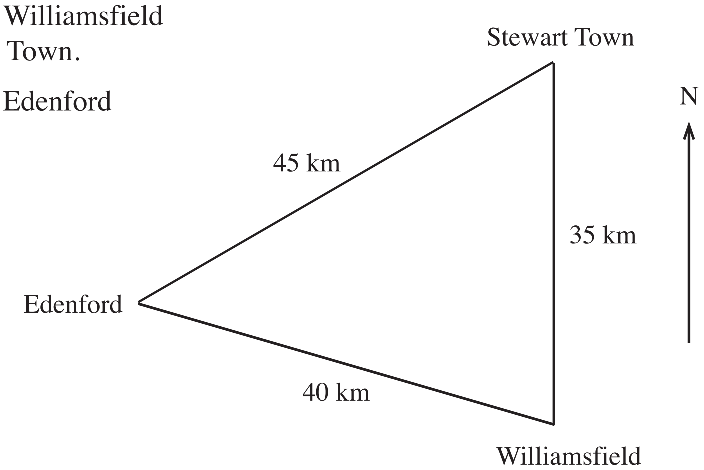
(b) Calculate the size of the angle NKB.



13. Stewart Town is 35 km due north of Williamsfield

Edenford is 40 km from Williamsfield and 45 km from Stewart Town.

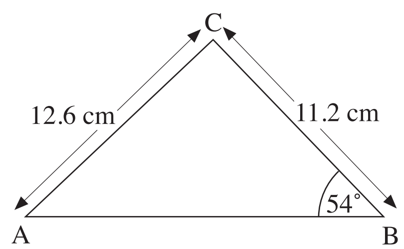
Calculate the bearing of Edenford from Williamsfield.



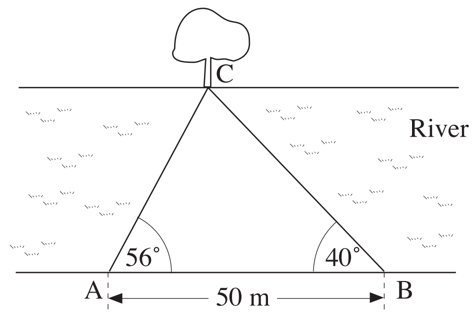
14. In triangle ABC, AC = 12.6 cm, BC = 11.2 cm and angle B = 54°. The lengths AC and BC are correct to the nearest millimetre and angle B is correct to the nearest degree. Use the sine rule

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

to calculate the smallest possible value of angle A.



15. The banks of a river are straight and parallel. To find the width of the river, two points, A and B, are chosen 50 metres apart. The angles made with a tree at C on the opposite bank are measured as angle CAB = 56°, angle CBA = 40°. Calculate the width of the river.

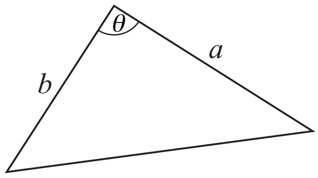


2 Application: Area of Any Triangle

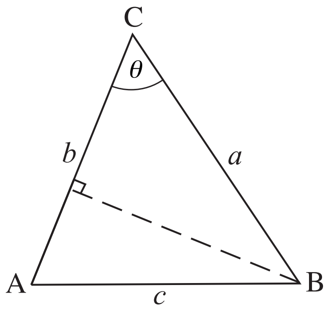
An important application of trigonometry is that of finding the area of a triangle with side lengths a and b and included angle θ.

The area (A) is given by $\frac{1}{2}ab\sin\theta°$.

$$A = \frac{1}{2}ab\sin\theta$$



Proof



If you construct the perpendicular from the vertex B to AC, then its length, p, is given by

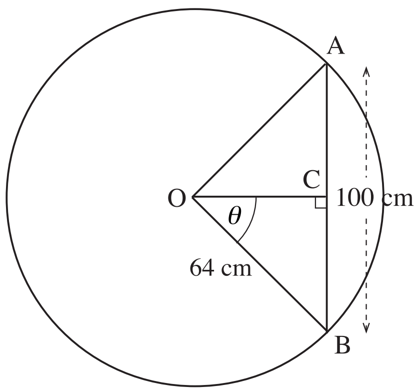
$$p = a\sin\theta$$

Thus the area of ABC is given by

$$\begin{aligned}\text{area} &= \frac{1}{2}\times\text{base}\times\text{height}\\ &= \frac{1}{2}\times b\times p\\ &= \frac{1}{2}\times b\times(a\sin\theta)\\ &= \frac{1}{2}ab\sin\theta\end{aligned}$$

as required.

Worked Example 1



The diagram shows a circle of radius 64 cm.

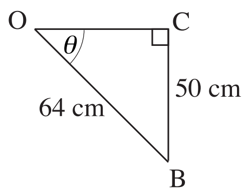
The length of the chord AB is 100 cm.

(a) Find the angle θ, to 2 d.p.

(b) Find the area of triangle OAB.

Solution

(a) If AB = 100 cm then, by symmetry, BC = 50 cm.



$$\sin\theta = \frac{50}{64}$$

$$\theta = 51.38°$$

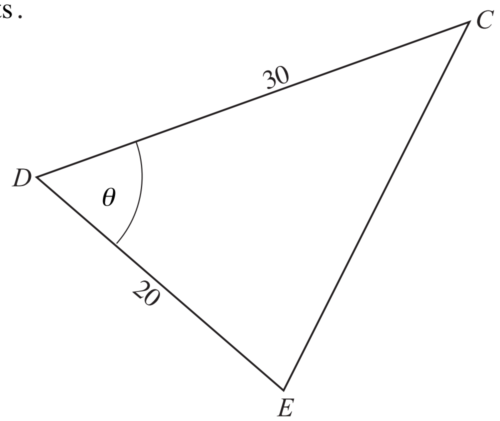
(b) The area of the triangle OAB is $\frac{1}{2}\times 64^2\times\sin 2\theta = 1997\text{ cm}^2$

Worked Example 2

Given that $\sin\theta = \dfrac{\sqrt{3}}{2}$, $0° \le \theta \le 90°$

(a) Express in fractional or surd form the value of $\cos\theta$.

(b) Show that the area of triangle *CDE* is $150\sqrt{3}$ square units, where $CD = 30$ units and $DE = 20$ units.



(c) Calculate the length of the side *EC*.

Solution

$$\sin^2\theta + \cos^2\theta = 1$$

so

$$\cos^2\theta = 1 - \left(\frac{\sqrt{3}}{2}\right)^2$$

$$= 1 - \frac{3}{4}$$

$$= \frac{1}{4}$$

Hence

$$\cos\theta = \frac{1}{2}$$

[An alternative approach is to consider the right angled triangle, as shown.

If the lengths

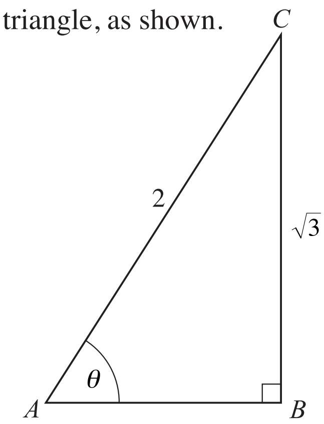
$$BC = \sqrt{3},\ AC = 2$$

then clearly $\sin\theta = \dfrac{\sqrt{3}}{2}$. We can find the length *AB*:

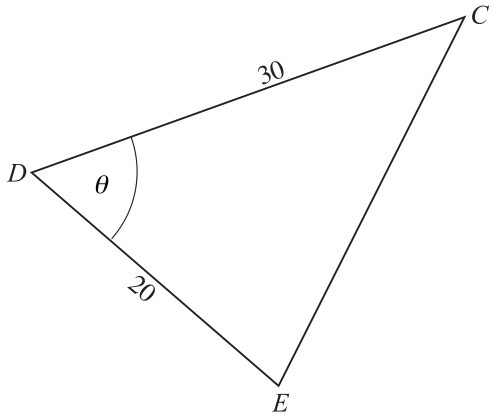
$$AB^2 = AC^2 - CB^2 \quad \text{(Pythagoras)}$$

$$= 4 - 3 = 1$$

So $AB = 1$ and $\cos\theta = \dfrac{AB}{AC} = \dfrac{1}{2}$]



(b) In the given triangle,



$$\text{Area} = \frac{1}{2} \times CD \times DE \sin\theta$$

$$= \frac{1}{2} \times 30 \times 20 \times \frac{\sqrt{3}}{2}$$

$$= 150\sqrt{3}$$

(c) We can find the length EC using the cosine rule, namely

$$EC^2 = DE^2 + DC^2 - 2 \times DE \times DC \times \cos\theta$$

$$= 20^2 + 30^2 - 2 \times 20 \times 30 \times \frac{1}{2}$$

$$= 400 + 900 - 600$$

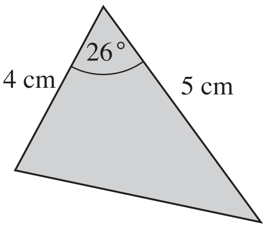
$$= 700$$

$$EC \approx 26.5$$

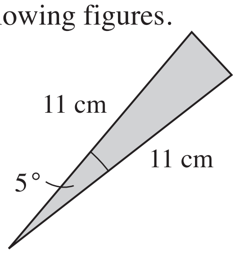
Exercises

1. Find the area of the shaded region in each of the following figures.

(a)

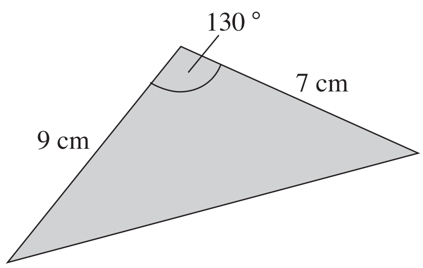


(b)

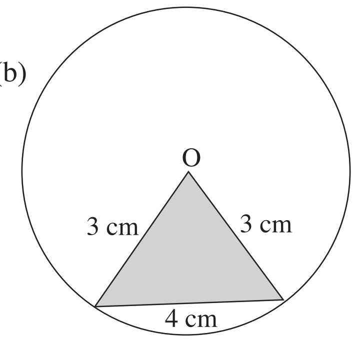


2. Find the area of the shaded region.

(a)

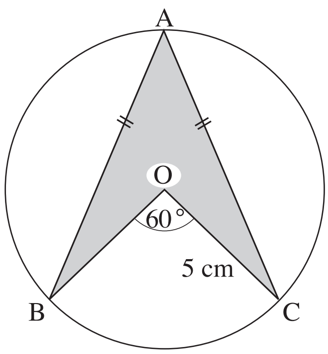


(b)



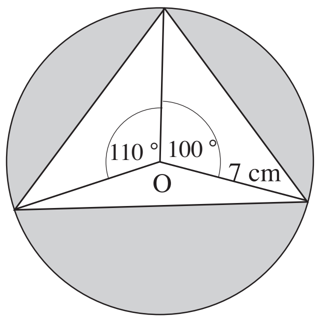
3. Find the area of the shaded region.

(a)

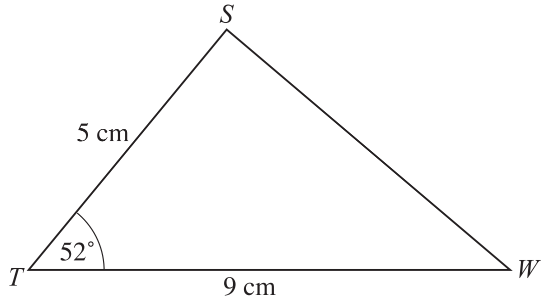


AB = AC

(b)



4.



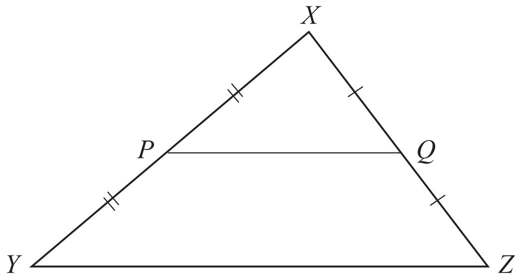
In the diagram above, **not drawn to scale**, $ST = 5$ cm , $TW = 9$ cm and $STW = 52°$.

Calculate

(a) the length of *SW*

(b) the area of ΔSTW.

5.

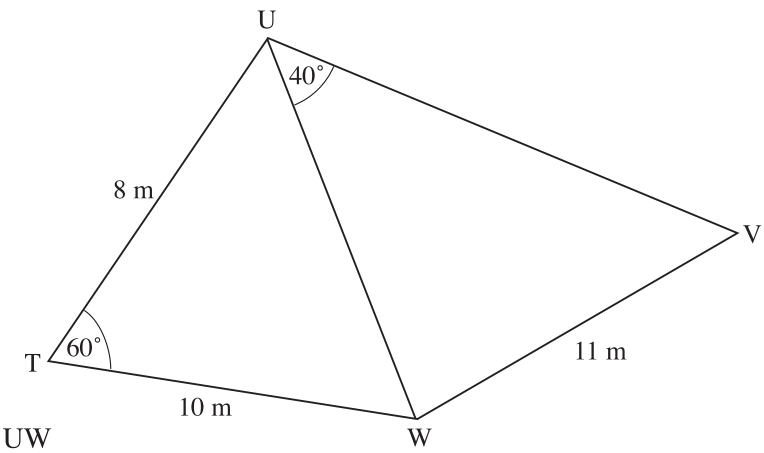


In the diagram above, **not drawn to scale**, *P* and *Q* are midpoints of the sides *XY* and *XZ* of triangle *XYZ.* Given that $XP = 7.5$ cm, $XQ = 4.5$ cm and the area of triangle $XPQ = 13.5\ \text{cm}^2$, calculate

(a) the size of angle *PXQ,* expressing your answer correct to the nearest degree.

(b) the area of triangle *YXZ.*

6. On the diagram below, **not drawn to scale**, TU = 8 m , TW = 10 m , VW = 11 m , angle UTW = 60° and angle WUV = 40°.



Calculate

(a) the length of UW

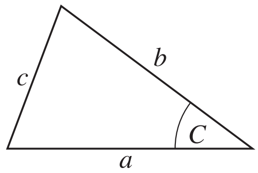
(b) the size of the angle UVW

(c) the area of triangle TUW.

3 Heron's Formula

You have already met the formula for the area of a triangle when the lengths of two sides and the included angle are known,

$$A = \frac{1}{2} a b \sin C$$



We will use this result to find the formula for the area of a triangle when the lengths of all three sides are known.

The formula is credited to Heron (or Hero) of Alexandria, and a proof can be found in his book, *Metrica,* written in about AD 60.

The formula, known as Heron's formula, is given by

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

where

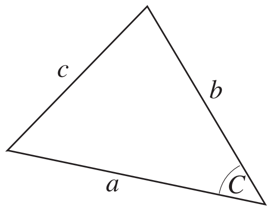
$$s = \frac{a+b+c}{2}$$

The proof is given below. Although straightforward, it does involve detailed algebraic manipulation.

Proof

You start with the formula

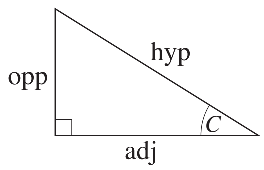
$$A = \frac{1}{2} a b \sin C$$



By definition, in a right angled triangle,

$$\sin C = \frac{\text{opp}}{\text{hyp}}$$

$$\Rightarrow \quad (\sin C)^2 = \frac{(\text{opp})^2}{(\text{hyp})^2}$$



$$= \frac{(\text{hyp})^2 - (\text{adj})^2}{(\text{hyp})^2} \quad \text{(using Pythagoras' theorem)}$$

$$= 1 - \frac{(\text{adj})^2}{(\text{hyp})^2}$$

$$= 1 - (\cos C)^2$$

3

Thus

$$A = \frac{1}{2}ab\sqrt{1-(\cos C)^2}$$

$$= \frac{1}{2}ab\sqrt{(1-\cos C)(1+\cos C)}$$

But, from the cosine rule (Unit 34),

$$c^2 = a^2 + b^2 - 2ab\cos C$$

i.e. $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$

Substituting into the formula for A,

$$A = \frac{1}{2}ab\sqrt{\left(1-\left(\frac{a^2+b^2-c^2}{2ab}\right)\right)\left(1+\left(\frac{a^2+b^2-c^2}{2ab}\right)\right)}$$

$$= \frac{1}{2}ab.\frac{1}{2ab}\sqrt{\left(2ab-a^2-b^2+c^2\right)\left(2ab+a^2+b^2-c^2\right)}$$

$$= \frac{1}{4}\sqrt{\left(c^2-(a-b)^2\right)\left((a+b)^2-c^2\right)}$$

$$= \frac{1}{4}\sqrt{\left(c-(a-b)\right)\left(c+(a-b)\right)\left((a+b)-c\right)\left((a+b)+c\right)}$$

$$= \frac{1}{4}\sqrt{(c+b-a)(c+a-b)(a+b-c)(a+b+c)}$$

But $\quad s = \frac{1}{2}(a+b+c)$

and

$$s-a = \frac{1}{2}(a+b+c)-a = \frac{1}{2}(b+c-a)$$

$$s-b = \frac{1}{2}(a+c-b)$$

$$s-c = \frac{1}{2}(a+b-c)$$

giving

$$A = \frac{1}{4}\sqrt{2(s-a)2(s-b)2(s-c)2s}$$

$$= \sqrt{s(s-a)(s-b)(s-c)}$$

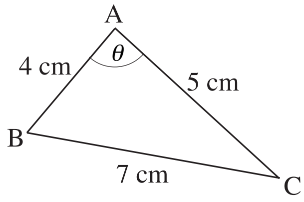
as required.

Worked Example 1

For the triangle shown, find

(a) the area of the triangle,

(b) angle θ.



Solution

(a) Using Heron's formula

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

where $$s = \frac{1}{2}(4+5+7) = 8 \text{ cm}$$

$$A = \sqrt{8 \times 4 \times 3 \times 1} = \sqrt{96} = 9.80 \text{ cm}^2$$

(b) Using the formula $A = \frac{1}{2}ab\sin\theta°$

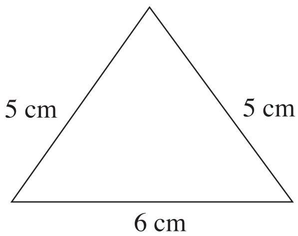
$$\sin\theta = \frac{2 \times A}{ab} = \frac{2 \times 9.8}{4 \times 5} = 0.98$$

$$\theta = 78.5°$$

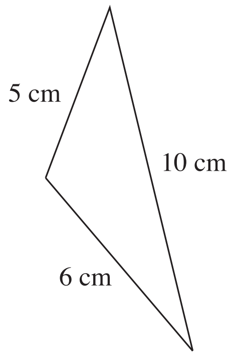
Exercises

1. Calculate the areas of the triangles shown.

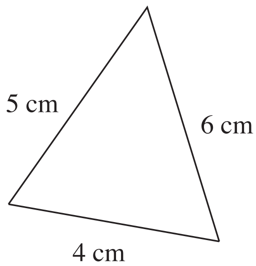
(a)



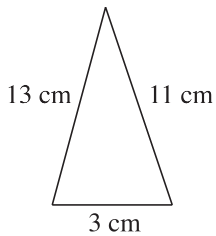
(b)



(c)



(d)



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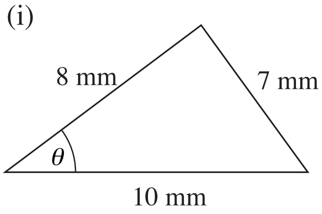
2. For each of the triangles shown find

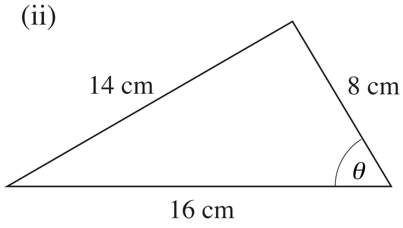
   (a) the area of the triangle,

   (b) the angle shown by θ.

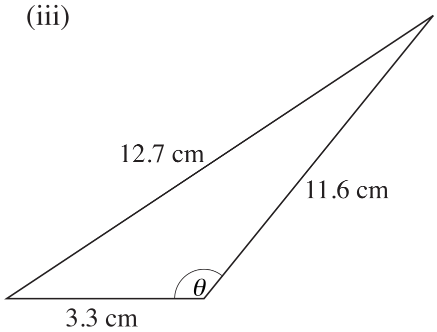
(i)



(ii)



(iii)



(iv)