Rates of Change

1 Introduction

Most things change: the thickness of the ozone layer is changing with time; the diameter of a metal ring changes with temperature; the air pressure on a mountain changes with altitude. In many cases, however, what is important is not whether things change, but how fast they change.

The study of rates of change has an important application, namely the process of **optimisation**. An example of an optimisation problem that you have already met is deciding what proportions a metal can should have in order to use the least material to enclose a given volume.

You may have seen a sign similar to the one below.

These are often put by the roadside to discourage drivers on main roads from driving too fast through small towns or villages; it is in such places that the police often set up 'speed traps' to monitor vehicle speeds and record information on those exceeding the speed limit.

Worked Example 1

The town of Dorchester in Dorset is 2 km from end to end, and a 30 mph speed limit is in force throughout. Although the A35 road now by-passes the town, many drivers consider it quicker, late at night when traffic is light, to drive through the centre.

1 km	= 0.6214 miles
1 mile	= 1.6093 km

(a) A driver took 2 minutes 40 seconds to drive through the town.
Was the speed limit broken?

(b) What is the shortest time a driver can take to drive through Dorchester and not break the speed limit?

Solution

(a) $$\text{Average speed} = \frac{\text{total distance}}{\text{time taken}} = \frac{2 \text{ km}}{160 \text{ sec}}$$

But $2 \text{ km} = 2 \times 0.6214 \text{ miles} \approx 1.243 \text{ miles}$

$$160 \text{ sec} = \frac{160}{60 \times 60} \text{ hours} \approx 0.0444 \text{ hours}$$

$$\text{giving average speed} \approx \frac{1.243}{0.0444} \approx 27.97 \text{ mph}$$

So it is possible that the speed limit was not broken.

(b) Travelling at a constant speed of 30 mph, the time taken is given by

$$\text{time taken} = \frac{1.243}{30} \text{ hours}$$

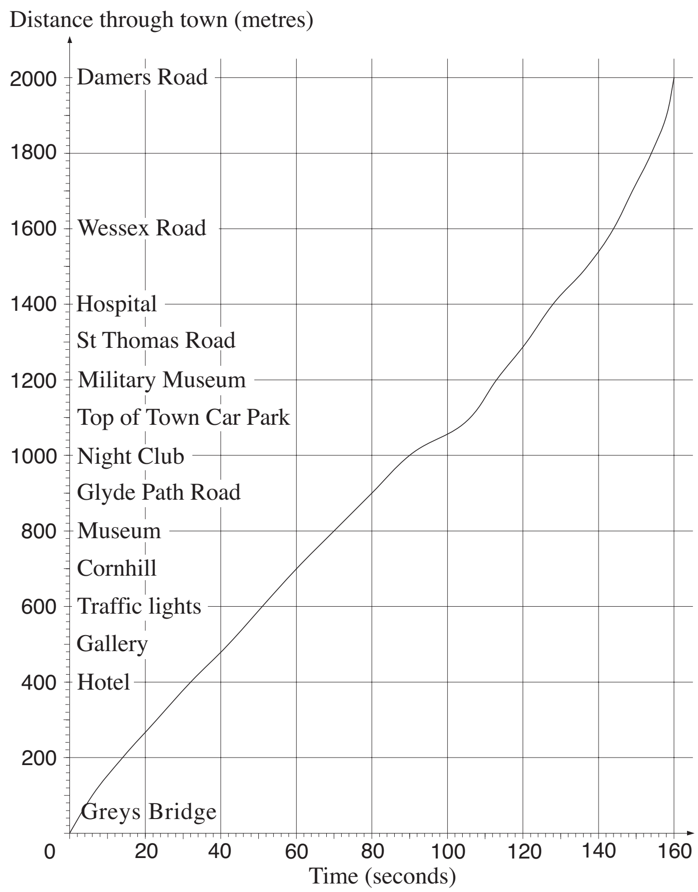
$$= \frac{1.243 \times 60}{30} \text{ min}$$

$$= 2.486 \text{ min or } 2 \text{ min } 29 \text{ sec.}$$

In reality cars do not travel at a constant speed. Suppose the driver's progress through the town was described by the distance-time graph in Worked Example 2.

Worked Example 2

The graph below is a distance-time graph for a car being driven through Dorchester.



(a) Travelling through Dorchester the driver negotiated a major roundabout. Where do you think it is, and how can you tell?

(b) What was the car's average speed, in mph, between

  (i) Grey's Bridge and the Night Club;

  (ii) the Night Club and the Hospital;

  (iii) Wessex Road and Damers Road?

Solution

(a) Roundabout is at Top of Town Car Park, as the gradient on the graph (i.e. speed) is reduced at that location.

(b) (i) Average speed $= \dfrac{\text{distance travelled}}{\text{time taken}} \approx \dfrac{1000}{90}$ m/s

$$= \frac{0.6214}{90 \div 3600} \text{ mph}$$

$$= 24.86 \text{ mph}$$

(ii) Average speed $\approx \dfrac{400}{38}$ m/s

$$\approx \frac{0.4 \times 0.6214}{38 \div 3600} \text{ m/s}$$

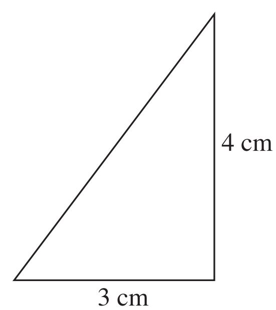
$$= 23.55 \text{ mph}$$

(iii) Average speed $\approx \dfrac{400}{15}$ m/s

$$= \frac{0.4 \times 0.6214}{15 \div 3600} \text{ mph}$$

$$= 59.65 \text{ mph}$$

Worked Example 3



(a) Construct a right-angled triangle, as shown in the diagram, and use it to represent a speed of 30 mph.

(b) When does the car in Worked Example 2 exceed the 30 mph speed limit?

(c) What would be the best place for the police to set a radar device to monitor motorists exceeding the speed limit?

(d) Assuming it was working accurately, what would the car's speedometer have shown as the car passed

(i) the Night Club;

(ii) Wessex Road?

Solution

(a) The speed is given by $\dfrac{4 \times 200}{3 \times 20}$ m/s

$$= \frac{40}{3} \text{ m/s}$$

$$= \frac{0.04}{3 \div 3600} \text{ km/h}$$

$$= 48 \text{ km/h}$$

$$\approx 48 \times 0.6214 \text{ mph}$$

$$= 29.8 \text{ mph}$$

$$\approx 30 \text{ mph}$$

(b) Between the Top of the Town Car Park and the Military Museum the car exceeds 30 mph; also from Wessex Road onwards it travels at over 30 mph.

(c) Between Wessex Road and Damers Road the speeds are clearly in excess of 30 mph so this would be a suitable place to monitor speeds.

(d) (i) Approximately 8 m/s $\approx \dfrac{0.008 \times 0.6214}{1 \div 3600} \approx$ 18 mph

(ii) Approximately 17.5 m/s $\approx \dfrac{0.0175 \times 0.6214}{1 \div 3600} \approx$ 39 mph

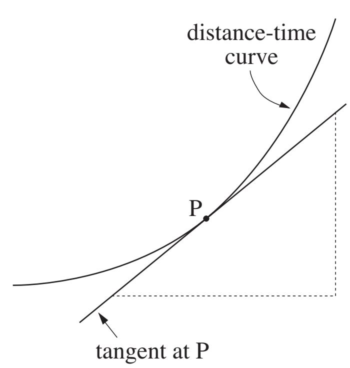
2 Instantaneous Speed

One method the police use to discover whether or not a car is speeding is to use video cameras to record the time taken to travel between two fixed points. In the example above, suppose the car was timed between Glyde Path Road and the Hospital; then the speed would have been calculated like this:

$$\frac{\text{distance travelled}}{\text{time taken}} = \frac{500}{50} = 10 \text{ ms}^{-1} \left(\text{about } 22\tfrac{1}{2} \text{ mph}\right)$$

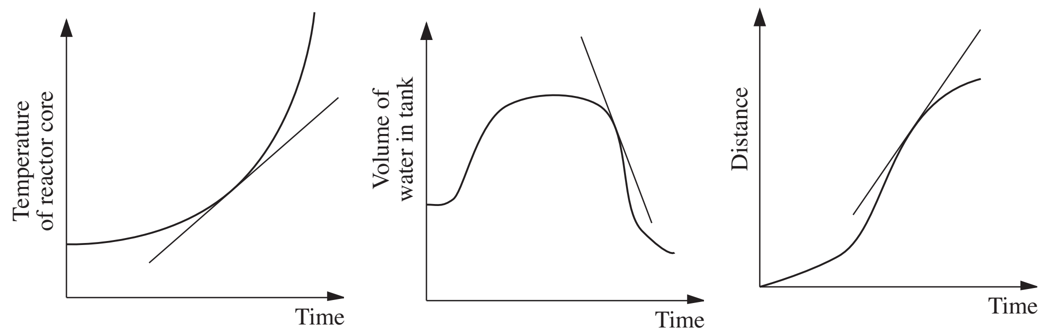
This figure is only an average speed. However the car's actual speed varied between these two points, and it may have gone faster than 30 mph and then slowed down.

Another method of finding the speed is to use a 'radar gun', which is focused on the car as it passes. This gives the **instantaneous** speed of the vehicle, as shown on the speedometer.

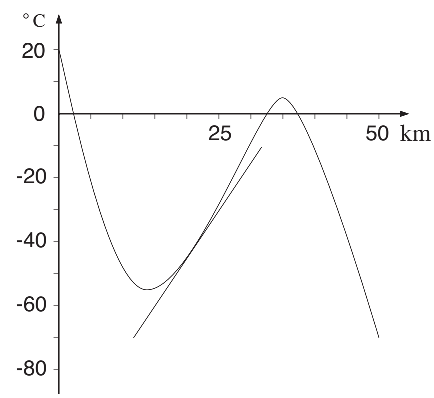


On a distance-time graph, the instantaneous speed is indicated by the steepness, or gradient, but when the graph is a complicated curve the gradient is difficult to pin down accurately. One way is to draw a tangent to the curve and to work out the gradient as in Worked Example 3 above.

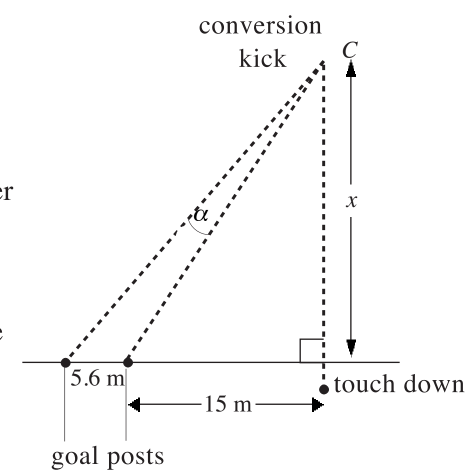
There are many other examples of a graph where the **gradient**, or steepness, have an important significance. This property of the gradient has important applications to all kinds of graphs, as illustrated below.



Rates of change have meaning even when neither of the variables is time. For example, the graph opposite shows how the temperature changes with height above sea level. The gradient of the **tangent** at P is $3°\text{C/km}$. This indicates that, 1 km above P, the temperature will be approximately $3°\text{C}$ higher. The rate of change of temperature with distance is sometimes called the **temperature gradient**.



In **rugby union** a try, worth 5 points, is scored by touching the ball down behind the line of the goal-posts. An extra 2 points can be scored by subsequently kicking the ball between the posts, over the cross-bar. This is known as a 'conversion': 5 points is converted into 7 points. The conversion kick must be taken as shown in the diagram, from somewhere on the line perpendicular to the goal line through the point where the ball was touched down.



Worked Example 1

Imagine that a try has been scored 15 m to one side of the goal. The angle in which the ball must be propelled is marked α; the smaller the angle, the trickier the kick. The angle depends on how far back the kick is taken. The way it changes is shown in the table.

Distance $(x \text{ m})$	0	5	10	15	20	25	30	35	40	45	50
Angle (α)	0	4.8	7.8	8.9	9.0	8.5	7.9	7.3	6.7	6.2	5.7

(a) Draw a graph of angle against distance.

(b) Estimate the gradient of the graph at $x = 5$ m and $x = 15$ m.

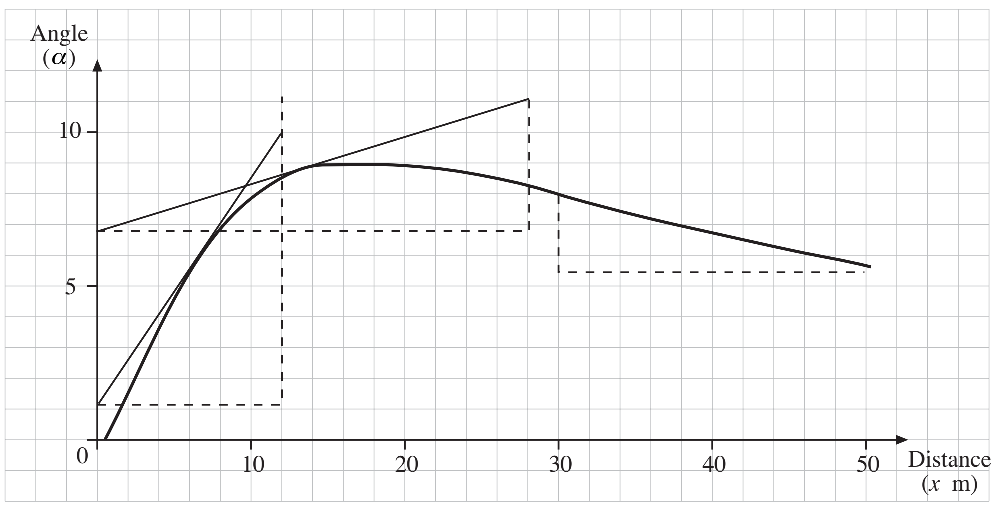
(c) What connection do the figures in (b) have with rates of change? What do these figures tell you?

(d) Estimate the gradient of the curve when $x = 40$ m. Interpret your answer.

(e) What is the best point from which to take the kick? What is the gradient at this point?

Solution

(a)



(b) At $x = 5$ m, gradient $\approx \dfrac{9.7 - 1.2}{12} \approx 0.7$

At $x = 15$ m gradient $\approx \dfrac{11.0 - 6.8}{28} \approx 0.15$

(c) They show that the rate of change is slowing down from $x = 5$ m to $x = 15$ m.

(d) At $x = 40$ m, gradient $\approx \dfrac{5.5 - 7.8}{20} \approx -0.1$; the rate of change is now negative.

(e) The best point is at about $\alpha = 18°$ where the gradient is zero. This gives a maximum value to the angle and so gives the kicker more chance of succeeding in kicking the ball between the goalposts.

Exercises

1. Harriet is a passenger in a car being driven along a motorway. She monitors the progress of the journey by counting the distance markers by the roadside. She writes down the distance travelled from the start every 5 minutes.

Time (minutes)	0	5	10	15	20	25	30	35	40	45	50	55	60
Distance travelled (miles)	0	6.4	13.1	20.0	26.2	31.3	35.5	39.2	43.4	47.9	53.1	59.8	67.0

(a) Draw a distance-time graph. Estimate the instantaneous speed of the car in mph after

(i) 20 mins (ii) 40 mins (iii) 55 mins

(b) The speed limit is 70 mph. Estimate from your graph the times at which the car was exceeding the speed limit.

2. The table below shows approximately how the world's population in millions increased between 1700 and 1980.

Year	1700	1720	1740	1760	1780	1800	1820	1840
Pop.	560	610	670	730	790	850	940	1050

Year	1860	1880	1900	1920	1940	1960	1980
Pop.	1170	1330	1550	1870	2270	3040	4480

(a) Draw a graph to represent these data. Draw tangents at the years 1750, 1850 and 1950 and measure their gradients.

(b) Explain what meaning can be attached to the gradients in (a) and write a brief account of what they show.

(c) Find the gradient at the year 1980 and use your answer to estimate the population in 1981.

3. The height h of a stone above the ground is given by the formula $h = 2 + 21t - 5t^2$, where h is measured in metres and t in seconds.

(a) Draw a graph of h against t, for values of t between 0 and 5.

(b) Estimate the velocity of the stone after 1, 2, 3 and 4 seconds. Make sure your method is clear.

(c) Use one of your answers to (b) to estimate the value of h when $t = 1.1$. Check the accuracy by substituting $t = 1.1$ into the original formula.

3 Finding the Gradient

Question 3 of the last section required you to draw the graph of the function $h = 2 + 21t - 5t^2$ and to find the gradient at certain points. If you compare your answers to someone else's you may well find that they do not agree precisely; this is because the process of drawing tangents is not a precise art - different peoples' tangents will have slightly different slopes. Moreover, the process of drawing and measuring tangents can become tiresome if repeated too often - you may well agree!

If the function had been $2 + 21t$ then finding the gradient would have been easy. The function $2 + 21t - 5t^2$ is not linear but there is more than one way of determining an accurate value for the gradient at any point, as you will see in the next worked example.

Worked Example 1

For the function $y = 2 + 21x - 5x^2$, find the values of y when $x = 0.95$ and $x = 1.05$.

Use these values to estimate the gradient at the point (1, 18).

Solution

$x = 0.95$, $y = 17.4375$

$x = 1.05$, $y = 18.5375$

Estimate of gradient at $x = 1$ is given by $\dfrac{18.5375 - 17.4375}{1.05 - 0.95}$

$$= \frac{1.1}{0.1}$$

$$= 11$$

The method developed in the example above has the advantage of accuracy, but it still takes time. A more efficient method is desirable, and the next two examples examine the simplest non-linear function of all with a view to finding one.

Worked Example 2

Use the method above to find the gradient of $y = x^2$ at different points, $x = 1, 2$ and 3. Make a table of the results and describe the pattern.

Solution

For $x = 1$, consider

$$x = 0.95 \Rightarrow y = 0.9025$$

$$x = 1.05 \Rightarrow y = 1.1025$$

$$\text{gradient estimate} = \frac{1.1025 - 0.9025}{1.05 - 0.95}$$

$$= \frac{0.2}{0.1}$$

$$= 2$$

(b) For $x = 2$, consider

$$\left.\begin{array}{l} x = 1.95 \Rightarrow y = 3.8025 \\ x = 2.05 \Rightarrow y = 4.2025 \end{array}\right\} \text{gradient estimate} = \frac{0.4}{0.1} = 4$$

(c) For $x = 3$

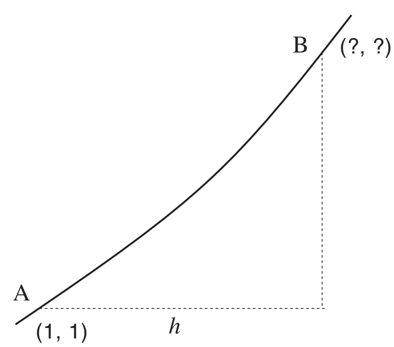
$$\left.\begin{array}{l} x = 2.95 \Rightarrow y = 8.7025 \\ x = 3.05 \Rightarrow y = 9.3025 \end{array}\right\} \text{gradient estimate} = \frac{0.6}{0.1} = 6$$

x	1	2	3
gradient	2	4	6

It appears that the gradient is twice the x value, that is, the gradient of x^2 is $2x$.

Worked Example 3

The diagram shows $y = x^2$ near the point (1, 1), labelled A. The point B is a horizontal distance h along from A.



(a) In terms of h, what are the coordinates of B?

(b) Find, and simplify as far as possible, a formula or the gradient of AB in terms of h.

(c) What happens to this formula as B gets closer to A?

(d) Repeat steps (a) to (c) for the points $(2, 4)$ and $(5, 25)$.

Solution

(a) The x value of B is $1 + h$, so the y value is $(1+h)^2$, i.e. coordinates $\left(1+h, (1+h)^2\right)$.

(b) Gradient of AB

$$= \frac{(1+h)^2 - 1}{(1+h) - 1}$$

$$= \frac{1 + 2h + h^2 - 1}{h}$$

$$= \frac{2h + h^2}{h}$$

$$= 2 + h$$

(c) As $B \to A$, $h \to 0$ and the gradient $\to 2$.

(d) For A being $(2, 4)$, the neighbouring point is $\left(2+h, (2+h)^2\right)$ giving

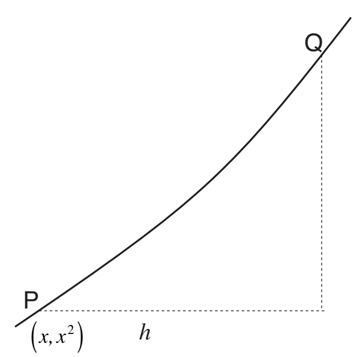
$$\text{gradient of AB} = \frac{(2+h)^2 - 4}{(2+h) - 2} = \frac{4h + h^2}{h} = 4 + h \to 4 \text{ as } h \to 0$$

For A being $(5, 25)$, B is $\left(5+h, (5+h)^2\right)$, giving

$$\text{gradient of AB} = \frac{(5+h)^2 - 25}{(5+h) - 5} = \frac{10h + h^2}{h} = 10 + h \to 10 \text{ as } h \to 0$$

The example above should have convinced you that at any point of $y = x^2$ the gradient is double the x-coordinate.

This procedure can be generalised. Suppose the point $(3, 9)$ is replaced by the general point $\left(x, x^2\right)$. In the diagram this point is denoted P, and Q has coordinates $(x+h, (x+h)^2)$.



The gradient of PQ is

$$\frac{(x+h)^2 - x^2}{h} = \frac{x^2 + 2hx + h^2 - x^2}{h}$$

$$= \frac{2hx + h^2}{h}$$

$$= \frac{h(2x+h)}{h}$$

$$= 2x + h \text{ (dividing by } h\text{)}$$

This shows that the gradient at (x, x^2) is $2x$, that is, double the x-coordinate.

Exercises

1. The diagram illustrates $y = x^3$ for $x > 0$.

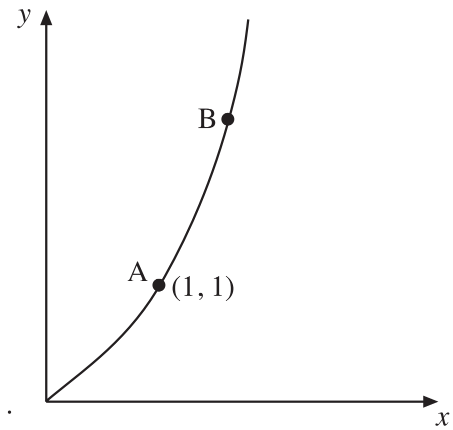
   The point (1, 1) is labelled A; point B has x coordinate $(1 + h)$.

   (a) What is the y coordinate of B?

   (b) Find a formula for the gradient AB in terms of h.

   (c) What happens at $h \to 0$?

   (d) Deduce the gradient of $y = x^3$ at $x = 1$.



2. In the same way as in Questions 1, deduce the gradient of $y = x^3$ at $x = 2$.

4 Gradient of Quadratics

The next worked example may take some time. Its purpose is to establish a formula for the gradient of any quadratic curve, that is, any curve with an equation of the form $y = ax^2 + bx + c$ where a, b and c are constants.

Worked Example 1

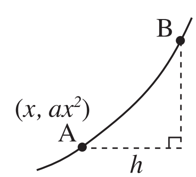
(a) Find the formula for the gradient of $y = ax^2$ where a is a constant.

(b) Find the formula for the gradient of $y = bx$ where b is a constant.

Solution

(a) Consider the neighbouring points with x coordinates x and $x + h$.

The corresponding values of y are ax^2 and $a(x + h)^2$, that is, A : (x, ax^2) and B : $(x + h, a(x + h)^2)$.



$$\text{So gradient of AB} = \frac{a(x + h)^2 - ax^2}{(x + h) - x}$$

$$= \frac{a(x^2 + 2xh + h^2) - ax^2}{h}$$

$$= \frac{2axh + ah^2}{h}$$

$$= 2ax + ah$$

As $h \to 0$, the gradient of AB becomes the gradient of the curve and has value $2ax$.

(b) Consider the neighbouring points with coordinates x and $x + h$ as in (a) above. Points A and B have coordinates A : (x, bx) B : $(x + h, b(x + h))$

and gradient of AB $= \dfrac{b(x+h) - bx}{(x+h) - x}$

$$= \frac{bh}{h}$$

$$= b$$

Clearly, as $h \to 0$, the gradient remains at value b.

At any point of the curve $y = ax^2 + bx + c$ the gradient is given by the function

$$\boxed{2ax + b}$$

A useful way to remember this rule is as follows :

- the gradient of $y = x^2$ is given by $2x$, so the gradient of $y = ax^2$ is $a \times 2x$
- the gradient of $y = bx$, a straight line, is just b;
- adding a constant, c, merely moves the curve up or down and does **not** alter the gradient;
- the formula $2ax + b$ comes from combining these properties.

Worked Example 2

Find formulae for the gradients of these curves :

(a) $y = 5x^2 - 7x + 10$ (b) $s = \dfrac{5}{8}t - \dfrac{1}{6}t^2$ (c) $q = \dfrac{2p^2 - 7p + 8}{3}$

Solution

(a) gradient $= 5 \times (2x) - 7 = 10x - 7$

(b) gradient $= \dfrac{5}{8} - \dfrac{1}{6} \times (2t) = \dfrac{5}{8} - \dfrac{1}{3}t$

(c) gradient $= \dfrac{2 \times (2p) - 7}{3}$

$$= \frac{4}{3}p - \frac{7}{3}$$

Worked Example 3

If $y = \dfrac{x^2 - 7x}{10} + 17$, what is the gradient when $x = 15$?

Solution

To answer this question, find the gradient function and then substitute the value 15 for x. Now

$$\text{gradient} = \frac{2x-7}{10}$$

When $x = 15$, gradient $= \dfrac{2 \times 15 - 7}{10} = \dfrac{23}{10} = 2.3$.

Exercises

1. Find formulae that give the gradients of these curves

(a) $y = 2x^2$

(b) $y = x^2 + x$

(c) $s = t^2 + 4t - 8$

(d) $y = x^2 - x + 10$

(e) $h = 6l^2 - 7$

(f) $y = 10x - \dfrac{x^2}{5}$

(g) $T = \dfrac{1}{9}Y^2 + 3Y - 1$

(h) $A = \dfrac{n^2 - 5n + 10}{2}$

(i) $u = v - 6v^2 + \dfrac{1}{15}$

(j) $y = \dfrac{3}{4}x^2 + \dfrac{x}{5} + 2$

2. Find the gradients of

(a) $y = 10 + 5x - 3x^2$ when $x = 2$

(b) $p = \dfrac{T^2}{2} + 8T - 16$ when $T = -3$

(c) $y = 3u^2 - \dfrac{u}{6}$ when $u = 4$

(d) $y = \dfrac{x^2 + 7x - 3}{12}$ when $x = 10$

(e) $m = 100 + 65N - \dfrac{N^2}{5}$ when $N = -15$

3. The gradient of the graph of $h = 2 + 21t - 5t^2$ gives the speed of a stone where h is the height in metres and t the time in seconds. Find the speed

(a) when $t = 0.5$

(b) when $t = 2.8$

5 Differentiation

The process of finding ‘gradient functions’ is called **differentiation**. Another name for the gradient function is the derivative or derived function. Hence the function

$3x^2 - 12x + 5$ is **differentiated** to give $6x - 12$

$6x - 12$ is the **derivative** of $3x^2 - 12x + 5$

The inventor of this technique is generally thought to have been the English physicist and mathematician *Sir Isaac Newton* (1643-1747), who developed it in order to explain the movement of stars and planets. However, the German mathematician *Gottfried Wilhelm Leibniz* (1646-1716) ran him close, and it was Leibniz who was the first actually to publish the idea, in the year 1684. Much vigorous and acrimonious discussion ensued as to who discovered the technique first. Today both are saluted for their genius.

The notation used by Leibniz is still used today. The gradient of a straight line is

$$\frac{\text{change in } y}{\text{change in } x}$$

which he shortened to $\frac{dy}{dx}$ (read as 'dy by dx').

The above example could be written thus:

$$y = 3x^2 - 12x + 5 \Rightarrow \frac{dy}{dx} = 6x - 12$$

or alternatively

$$\frac{d}{dx}(3x^2 - 12x + 5) = 6x - 12$$

the symbol $\frac{'d'}{dx}$ standing for the derivative with respect to x.

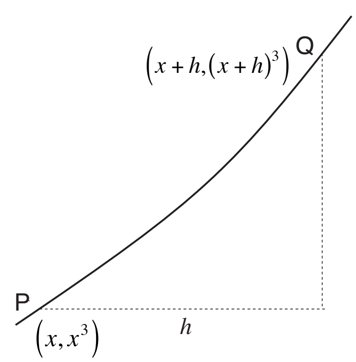
Another way of denoting a derived function is to use the symbol f', as follows:

$f(x) = 3x^2 - 12x + 5$ (function)

$f'(x) = 6x - 12$ (derived function)

We now consider the derivative of the function x^3.

With reference to the diagram opposite, the gradient of PQ is given by



$$\frac{(x+h)^3 - x^3}{h}$$

$$= \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$

$$= 3x^2 + 3xh + h^2$$

Whatever the value of x, this gradient gets closer and closer to $3x^2$ as $h \to 0$, so

$$\frac{dy}{dx} = 3x^2$$

Worked Example 1

Find the derivative of $y = \frac{1}{x}$ $(x \neq 0)$.

Solution

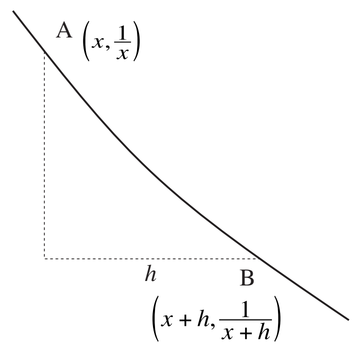
In this case the gradient of AB is

$$\left(\frac{1}{x+h} - \frac{1}{x}\right)\frac{1}{h}$$

$$= \left(\frac{x-(x+h)}{(x+h)x}\right)\frac{1}{h}$$

$$= \left(\frac{-h}{x(x+h)}\right)\frac{1}{h}$$

$$= \frac{-1}{x(x+h)}$$

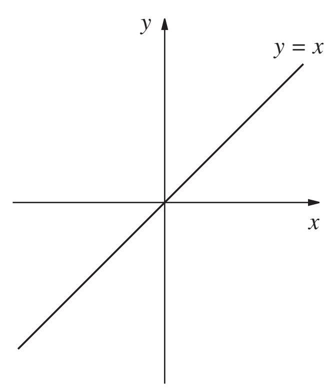


As h gets closer to zero, this formula gets closer to $\frac{-1}{x^2}$. Hence

$$\frac{dy}{dx} = -\frac{1}{x^2} \; (x \neq 0)$$

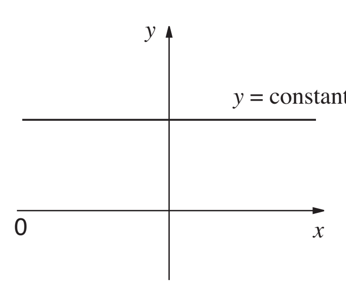
The derivatives of the functions x^2, x^3 and $\frac{1}{x}$ have now been established. Two other functions can be added to those, for completeness:

- the line $y = x$ has gradient 1, and so the derivative of the function x is 1;
- the line $y =$ constant has zero gradient, and so the derivative of a constant is 0.



A summary of the results obtained so far is as follows :

Function	**Derivative**
constant	0
x	1
x^2	$2x$
x^3	$3x^2$
$\frac{1}{x}$	$-\frac{1}{x^2}$



You may be able to guess the derivatives of higher powers of x; this and other matters will be covered in the last section of this unit.

In Section 4.3, it was observed that the derivative of, for example, $5x^2$ was 5 times the derivative of x^2.

Similarly, the derivative of $5x^3$ is 5 times the derivative of x^3 :

$$\frac{d}{dx}(5x^3) = 5 \times \frac{d}{dx}\left(x^3\right)$$

$$= 5 \times 3x^2$$

$$= 15x^2$$

Another assumption to make is that functions such as $x^2 + \frac{1}{x}$ can be differentiated by adding together the derivatives of x^2 and $\frac{1}{x}$:

e.g. $\frac{d}{dx}\left(x^2 + \frac{1}{x}\right) = \frac{d}{dx}(x^2) + \frac{d}{dx}\left(\frac{1}{x}\right) = 2x - \frac{1}{x^2}$

Worked Example 2

Differentiate the following functions :

(a) $y = 3x^3 - 5x + 6$ with respect to x

(b) $y = x(x-3)(x+4)$ with respect to x

(c) $A = 10q^3 - \frac{5}{q}$ with respect to q

(d) $P = \frac{(h^3 + 3)}{2h}$ with respect to h

Solution

(a) $\frac{dy}{dx} = 3\frac{d}{dx}\left(x^3\right) - 5\frac{d}{dx}(x) + \frac{d}{dx}(6) = 3(3x^2) - 5 = 9x^2 - 5$

(b) $y = x^3 + x^2 - 12x$ (brackets must first be multiplied out)

$\frac{dy}{dx} = 3x^2 + 2x - 12$

(c) $\frac{dA}{dq} = 10(3q^2) - 5(-\frac{1}{q^2})$ (note $\frac{dA}{dq}$ instead of $\frac{dy}{dx}$ as the derivation is of A with respect to q)

(d) $P = \frac{h^2}{2} + \frac{3}{2h}$ (function must be divided out)

$$\frac{dP}{dh} = \frac{2h}{2} + \frac{3}{2}\left(-\frac{1}{h^2}\right) = h - \frac{3}{2h^2}$$

Exercises

1. Find the derivative of the following functions :

   (a) $y = x^3 + 5x^2 + 3x$ with respect to x

   (b) $r = 6t^3 - 10t^2 + 2t$ with respect to t

   (c) $f(x) = 5x^2 + \frac{1}{x}$ with respect to x

   (d) $f(x) = x^2\left(x - \frac{1}{x}\right)$ with respect to x

   (e) $f(t) = \frac{t^3 + 3t}{5}$ with respect to t.

2. Differentiate these functions :

   (a) $(x+2)^2$ (b) $x(x+1)(x-1)$ (c) $s(s+\frac{1}{3})^2$

   (d) $\frac{8y^3 + 3y^2}{9} + 3$ (e) $\frac{x^4 - 5x^2 - 1}{x}$

3. (a) What is the gradient of the curve $y = x^3 - 3x^2 + 6$ at the point (3, 6)?

   (b) What is the gradient of the curve $y = 2x - \frac{5}{x}$ at the point (2, 1)?

   (c) At what point is the gradient of $y = x^2 + 6x + 3$ equal to 10?

   (d) When is the tangent to the curve $y = 3x^2 - 5x + 10$ parallel to the line $y = 20 - 11x$?

   (e) At what two points is the gradient of $y = 2x^3 - 9x^2 + 36x - 11$ equal to 24?

4. A student suggests that the height of the average male (beyond the age of 3) can be modelled according to the formula

$$h = 6 - \frac{12}{y}$$

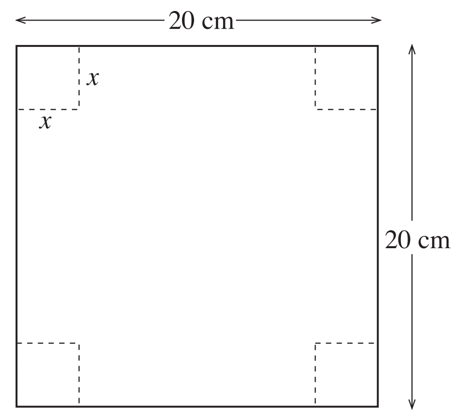
   where h is the height in feet and y is the age in years.

   Use this model to find the rate of growth of the average male (in feet per year) at the ages of

   (a) 6 years (b) 8 years

6 Optimisation

A piece of card 20 cm by 20 cm has four identical square pieces of side x removed from the corners so that it forms a net for an open-topped box. The problem is to find the dimensions of a box with the largest volume.



Worked Example 1

(a) Write down a formula for the volume V of the box described above in terms of x.

(b) Sketch a graph of V against x for all the allowable values of x.

(c) Find the gradient of the graph when $x = 1$, 2 and 3. Interpret these figures, in terms of rates of change.

(d) What is the gradient when $x = 4$? Interpret your answer.

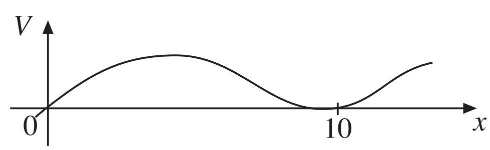
(e) Find the coordinates (x, V) where the gradient is zero. What is the significance of this?

Solution

(a) $V = (20 - 2x)(20 - 2x)\,x$

$V = 4x(10 - x)^2$

(b)



(c) $V = 4x(100 - 20x + x^2)$

$= 400x - 80x^2 + 4x^3$

Gradient $= 400 - 160x + 12x^2$

At $x = 1$, gradient $= 252$

$x = 2$, gradient $= 128$

$x = 3$, gradient $= 28$

The gradient is decreasing as x increases, i.e. the rate of increase in V is decreasing.

(d) At $x = 4$, gradient $= -48$, so now the V is decreasing.

(e) Gradient $= 0$ gives

$$0 = 400 - 160x + 12x^2$$

or $\quad 3x^2 - 40x + 100 = 0$

This is a quadratic, and can be factorised to give

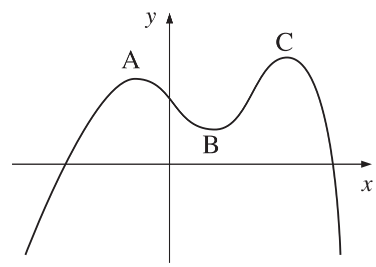
$$(3x - 10)(x - 10) = 0$$

giving $3x - 10 = 0$ or $x - 10 = 0 \Rightarrow x = \frac{10}{3}$ or 10

The points $x = \frac{10}{3}$ and $x = 10$ correspond to maximum volume and minimum volume (zero).

Worked Example 2

The graph opposite shows a sketch of the function $f(x)$. What is the value of $f'(x)$ at the points A, B and C ?

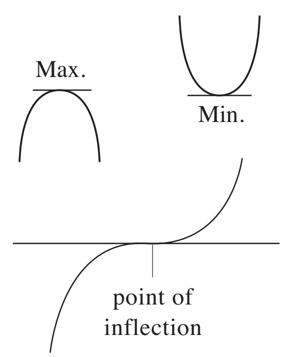


Solution

At points A, B and C the value of $f'(x)$ is zero.

The graph in the last example contained three examples of stationary points. This is the general term used to describe maximum and minimum points. At a stationary point the gradient of the graph is zero; the tangent is exactly horizontal.

The **maximum** and **minimum** points of any function can be found by working out where the gradient is zero. The process of finding maximum and minimum points is sometimes called **optimisation**. It should also be noted that stationary points can also turn out to be points of inflection, as illustrated opposite.



Worked Example 3

Find the largest volume of an open top box that can be made from a piece of A4 paper (20.9 cm by 29.6 cm).

Solution

Suppose squares of side x are cut from each corner. Then the volume is given by

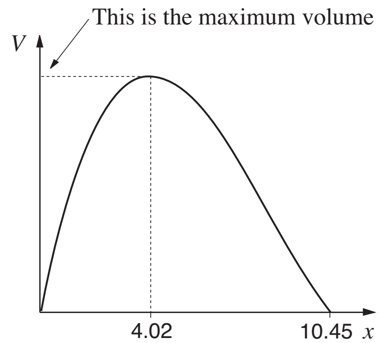
$$V = x(20.9 - 2x)(29.6 - 2x)$$

$$= 618.64x - 101x^2 + 4x^3$$

(Remember: brackets must be multiplied out before differentiation).

The volume is a maximum when the gradient is zero.

$$\frac{dV}{dx} = 618.64 - 202x + 12x^2$$



The required value of x can be obtained by solving the quadratic equation

$$12x^2 - 202x + 618.64 = 0$$

Hence $$x = \frac{202 \pm \sqrt{202^2 - 4 \times 12 \times 618.64}}{24}$$

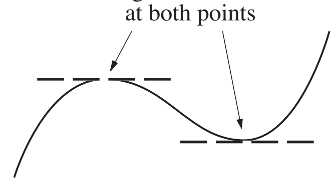
$$= 4.02 \text{ cm or } 12.8 \text{ cm}$$

12.8 cm is clearly inappropriate to this problem. Hence $x = 4.02$ cm is the size of square that maximises the volume. The largest volume is therefore the value of V when $x = 4.02$:

$$V_{max} = 4.02(20.9 - 2 \times 4.02)(29.6 - 2 \times 4.02)$$

$$= 1115 \text{ cm}^3 \text{ (to the nearest whole number)}$$

A potential snag with this method is that it only tells you where the stationary points are, but does not distinguish between maxima and minima. There are two simple ways round this problem.



Worked Example 4

Find the two stationary points of the function

$$T = 2k + \frac{8}{k}$$

and determine which is a maximum and which is a minimum.

Solution

Now $\frac{dT}{dk} = 0$ for stationary point,

and $$\frac{dT}{dk} = 2 - \frac{8}{k^2}$$

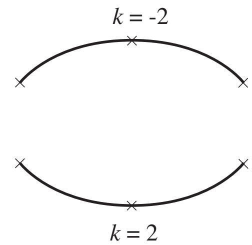
$$\Rightarrow \quad 2 - \frac{8}{k^2} = 0$$

$$\Rightarrow \quad 2 = \frac{8}{k^2}$$

$$\Rightarrow \quad k^2 = 4$$

$$\Rightarrow \quad k = 2 \text{ or } -2$$

When $k = 2,\ T = 8$ and when $k = -2,\ T = -8$.



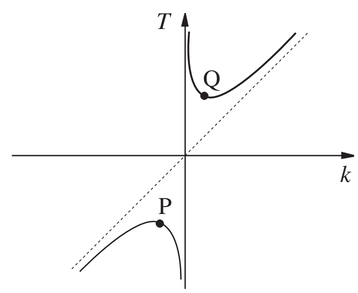
k	-2.1	-2	-1.9	k	1.9	2	2.1
T	-8.01	-8	-8.01	T	8.01	8	8.01
		maximum				minimum	

Hence the function has a

maximum at $(-2, -8)$

minimum at $(2, 8)$

Two important points arise from the last worked example :



1. Note that the maximum point is lower than the minimum.
2. The word 'maximum' is always taken to mean 'local maximum'. In the diagram, P is higher than any neighbouring point, but there are other points on the curve that are higher. Similarly, the word 'minimum' is taken to mean 'local minimum'.

Exercises

1. A function $f(x)$ is defined as follows:

$$f(x) = x^3 - 6x^2 - 36x + 15$$

Show that $f'(-2) = f'(6) = 0$, and hence find the coordinates of the maximum and minimum points.

2. Find the maximum and minimum points of these curves :

(a) $y = 2x^2 - 6x + 7$ (b) $y = 3x + \dfrac{27}{x}$

(c) $y = 70 + 105x - 3x^2 - x^3$ (d) $y = x^2 + \dfrac{16}{x}$

3. A manufacturing company has a total cost function

$$C = 5Q^2 + 180Q + 12500$$

This gives the total cost of producing Q units.

(a) Find a formula for the unit cost U, in terms of Q, where $U = C/Q$.

(b) Find the value of Q that minimises the unit cost. Find this minimum unit cost.

4. The makers of a car use the following polynomial model to express the petrol consumption M miles per gallon in terms of the speed v miles per hour,

$$M = \frac{v^3 - 230v^2 + 15100v - 145000}{4000}$$

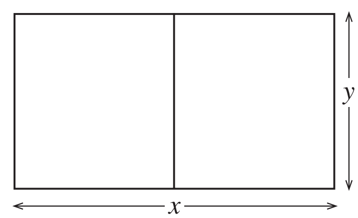
(a) Find the speed that maximises the petrol consumption, M.

(b) The manufacturers only use this model for $30 < v < 90$.
Do you think that this restriction is sensible?

7 Real Problems

Worked Example 1

You have 120 m of fencing and want to make two enclosures as shown in the diagram. The problem is to maximise the area, A, enclosed.



(a) To find the maximum area, differentiate the expression for A and put it equal to zero. What is the problem with doing this?

(b) Use the fact that the total length of fencing is 120 m to write an equation connecting x and y.

(c) Make y the subject of this equation. Hence write a formula for A in terms only of x. Now differentiate with respect to x to solve the original problem.

Solution

(a) The area, A, is given by $A = xy$, but x and y are connected by $120 = 3y + 2x$.

Hence $\quad 3y = 120 - 2x \Rightarrow y = \frac{1}{3}(120 - 2x)$

giving $\quad A = x \cdot \frac{1}{3}(120 - 2x)$

$$= 40x - \frac{2}{3}x^2$$

To find the maximum area, we find the gradient (or differential),

that is, $\quad \text{gradient} = 40 - \frac{4x}{3}$

and this is zero where

$$0 = 40 - \frac{4x}{3}$$

that is, $\quad 40 = \frac{4x}{3} \Rightarrow x = 30 \text{ m}$

When $x = 30$, $120 = 3y + 60$

$$60 = 3y$$

i.e. $y = 20$ m

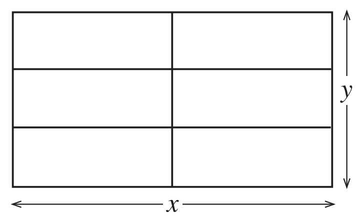
and the area, A, is given by

$$A = 30 \times 20 = 600 \text{ m}^2$$

The problem posed in the example above is different from those earlier in this section. The quantity that needed to be maximised was first expressed in terms of two quantities, x and y. However, x and y were connected by the condition that the total length of fencing had to be 120 m. This sort of condition is known as a constraint. It allowed area A to be expressed in terms of one quantity only, and thus the problem could be solved.

Worked Example 2

Find the maximum area that can be enclosed by 120 m of fencing arranged in the configuration on the right.



Solution

Let the overall dimensions be x metres and y metres and the area be A square metres.

$A = xy$ (the quantity to be maximised)

$4x + 3y = 120$ (constraint from total length of fencing)

$y = \dfrac{120 - 4x}{3}$ (make y the subject)

$$A = \frac{x(120 - 4x)}{3}$$

$$= 40x - \frac{4}{3}x^2$$

$$\frac{dA}{dx} = 40 - \frac{8}{3}x$$

At a stationary point $\dfrac{dA}{dx}$ must be zero; this gives

$$40 - \frac{8}{3}x = 0$$

$$\Rightarrow \quad x = 15$$

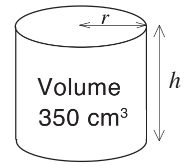
The question asked for the maximum area. From the equation for y

$$y = \frac{120 - 4 \times 15}{3} = 20$$

So maximum area $= 20 \times 15 = 300 \text{ m}^2$

Worked Example 3

A closed cylindrical can has a volume of 350 cm^3. Find the dimensions of the can that minimise the surface area.



Solution

Let the radius be r cm and the height h cm. Let the surface area be S cm^2; then

$$S = 2\pi r^2 + 2\pi rh \quad \text{(the quantity to be minimised)}$$

At present, S involves two variables, r and h. The fact that the volume has to be 350 cm^3 gives a connection between r and h; namely

$$\pi r^2 h = 350 \text{ (constraint)}$$

So $\quad h = \dfrac{350}{\pi r^2}$ (make h the subject)

and $\quad S = 2\pi r^2 + 2\pi r\left(\dfrac{350}{\pi r^2}\right)$ (substitute for h in the S formula)

$$= 2\pi r^2 + \frac{700}{r}$$

giving $\quad \dfrac{dS}{dr} = 4\pi r - \dfrac{700}{r^2}.$

At a stationary point, $\dfrac{dS}{dr} = 0$,

giving

$$4\pi r - \frac{700}{r^2} = 0$$

$$\Rightarrow \quad 4\pi r = \frac{700}{r^2}$$

$$\Rightarrow \quad r^3 = \frac{700}{4\pi} \approx 55.7$$

$$\Rightarrow \quad r = 3.82 \text{ cm to 3 s.f.}$$

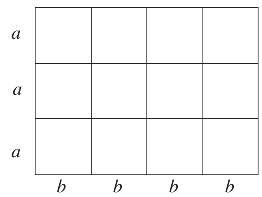
$$\Rightarrow \quad h = \frac{350}{\pi r^2} = 7.64 \text{ cm to 3 s.f. (from equation above)}$$

Note that this implies that

$$\text{diameter} = 2r \approx 7.64 \text{ cm} = \text{height}$$

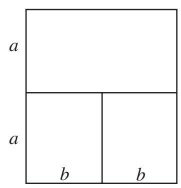
Exercises

1. The rectangular window frame in the diagram uses 20 m of window frame altogether. What is the maximum area the window can have?

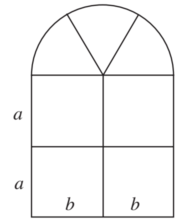


2. Repeat Question 1 for these window designs.

(a)

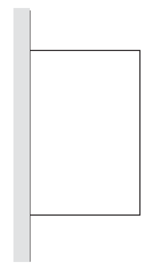


(b)



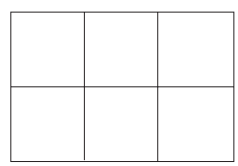
3. A rectangular paddock is to have an area of 50 m^2. One side of the rectangle is a straight wall; the remaining three sides are to be made from wire fencing.

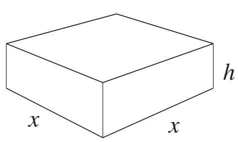
   What is the least amount of fencing required?



4. The enclosure shown has a total area of 300 m^2.

   Find the minimum amount of fencing required.



5. 

   A small closed water tank is in the shape of a cuboid with a square base. The total surface area is 15000 cm^2. The problem here is to maximise the volume.

   (a) Let x cm be the side of the square and h cm be the height. Write down an expression for the volume V.

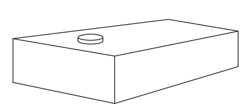
   (b) Show that $h = \dfrac{3750}{x} - \dfrac{x}{2}$

   (c) Show that the maximum volume is 125 litres.

6. An emergency petrol tank is designed to carry 1 gallon of petrol (4546 cm^3). Its shape can be considered to be a cuboid.

   The base of the cuboid is a rectangle with the length double the width.

   Find the dimensions of the tank that minimise the surface area required. Give the answers to the nearest millimetre.



7. The solution to the last worked example was such that the diameter and the height were equal. Show that this is true for any fixed volume of a cylinder when the surface area is to be minimised.

8. A particle is moving along a straight line. At time t seconds its distance s metres from a fixed point F is given by

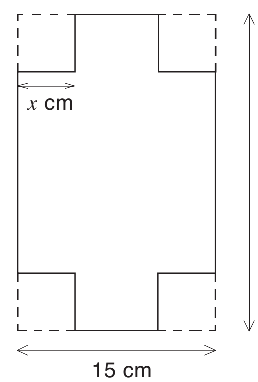
$$s = t^3 - 12t^2 + 45t + 10$$

(a) Its velocity v in ms^{-1} can be obtained by differentiating s with respect to t. Find v in terms of t.

(b) Find the two values of t for which the particle is stationary.

(c) The acceleration of the particle can be obtained by differentiating v with respect to t. When is the particle's acceleration zero?

9. The diagram on the right shows a 24 cm by 15 cm sheet of cardboard from which a square of side x cm has been removed from each corner.



The cardboard is then folded to form an open rectangular box of depth x cm and volume V cm^3. Show that

$$V = 4x^3 - 78x^2 + 360x$$

Find the value of x for which V is a maximum, showing clearly that this value gives a maximum and not a minimum value for V.