2

Exercises

1. Use the population model

$$P(t) = P_0 a^t$$

with $P_0 = 68$ million and

(a) $a = 1.01$ (b) $a = 1.05$ (c) $a = 1.1$

In each case, estimate when the population will reach 80 million.

2. In the question above, what factors in the population model have not been fully taken into account?

3 Carbon Dating

Carbon 14 (^{14}C) is an isotope of carbon with a half life of 5730 years. It exists in the carbon dioxide in the atmosphere, and all living things absorb some Carbon 14 as they breathe. This remains in an animal or plant, and is constantly added to until the organism dies. After this time, the Carbon 14 decays, reducing to half the amount stored in the body after 5730 years. The amount halves again after another 5730 years, and so on, with no new Carbon 14 absorbed.

In 1946 an American scientist, *Williard Libby*, developed a way of 'dating' archaeological objects by measuring the Carbon 14 radiation present in them. This radioactivity is compared with that found in things living now.

For instance, if bones of recently dead animals produce 10 becquerels per gram of bone carbon (a becquerel is the unit of radioactivity), and an old bone produces only 5 becquerels, the radioactivity has halved since the animal which had the old bone died. As the half life of Carbon 14 is 5730 years, this would mean the animal died in 3715 BC approximately, as tested in 2015.

Exercises

1. Complete the table below.

Age (in years)	*Radiation* (becs)
0	10
5730	$10 \times \frac{1}{2} = 10 \times 2^{-1}$
11460	$10 \times (\frac{1}{2})^2 = 10 \times 2^{-2}$
17190	...
22920	...
...	...

3

2. Use your model to produce a graph, showing radioactivity on the vertical axis and time, in years, on the horizontal. Draw the graph for values of t up to 50 000 years.

3. From your graph, estimate the ages of bones with these radioactivities:

   (a) 8.5 becquerels per gram of carbon;

   (b) 1.2 becquerels per gram of carbon.

4 Rate of Growth

Suppose a colony of bacteria doubles in number every minute as every member of the colony divides in two. So if there are 2 bacteria at the start of the colony, there will be 4 a minute later (an increase of 2 in one minute), 8 two minutes later (an increase of 4 in one minute) and so on. As the number of bacteria increases, so the **rate** at which that number increases goes up. So the rate of increase of an exponential function is closely related to the value of the function at any point. This suggests that exponential functions and their derivatives are closely linked.

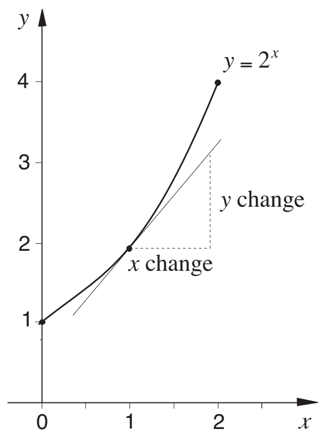
The next example will explore these links.

Worked Example 1

Plot and draw the curves below for values of x between 0 and +2 and on separate axes.

(a) $y = 2^x$ (b) $y = 3^x$

Using a ruler to draw tangents to each of the curves, calculate the gradient of each curve at $x = 0, 0.5, 1, 1.5$ and 2. The figure opposite illustrates the method.



Note that

$$\text{gradient} = \frac{\text{change in } y}{\text{change in } x}$$

In this case all gradients are positive since a positive change in x results in a positive change in y.

Plot the values for the gradient of each of the graphs and sketch in the gradient curve.

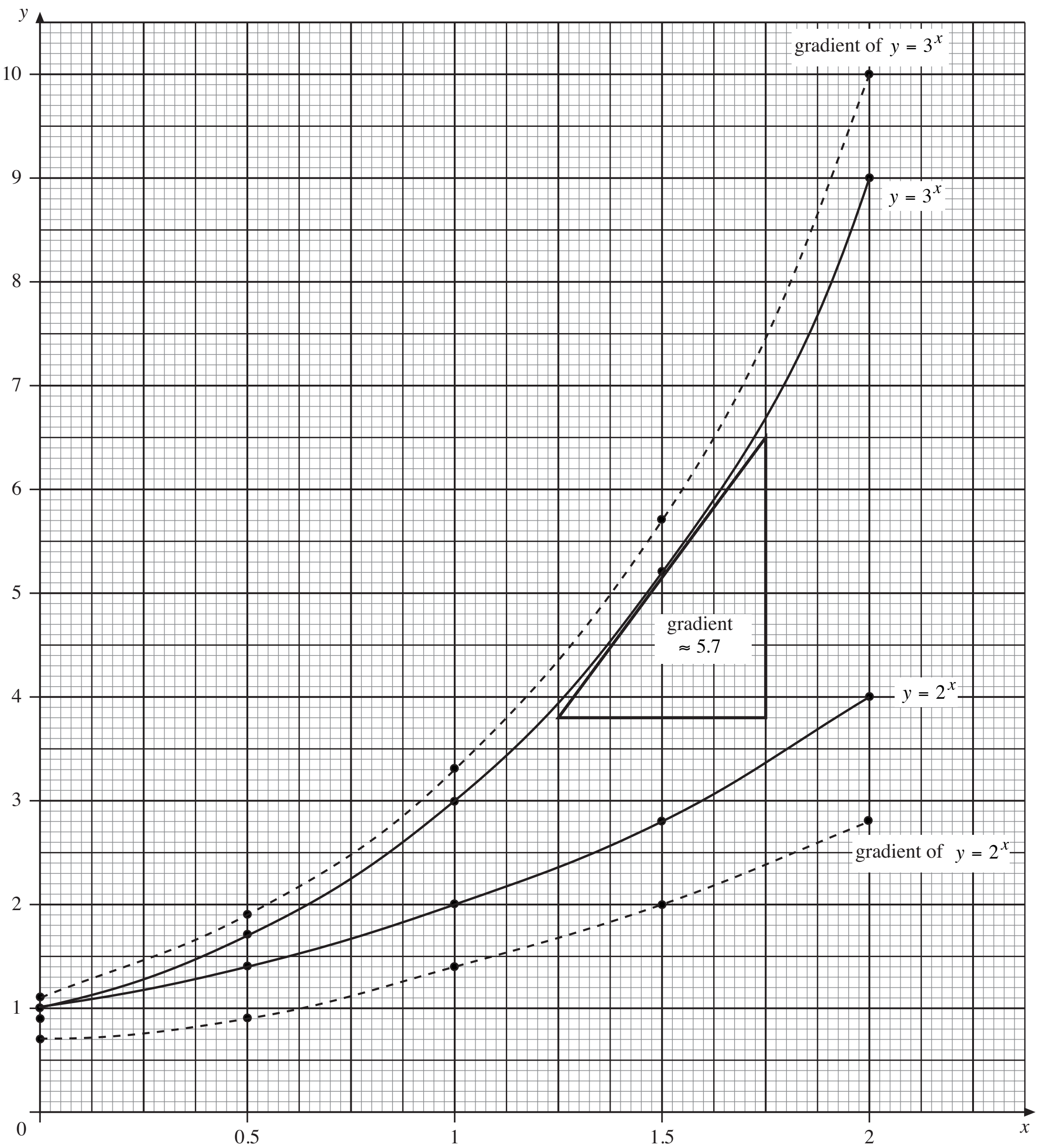
How do the original graph and the gradient curve of each function seem to be related?

Solution

The graphs are shown below with dashed curves to indicate the gradient curve.

x	0	0.5	1.0	1.5	2
2^x	1	1.41	2.0	2.83	4
3^x	1	1.73	3.0	5.20	9

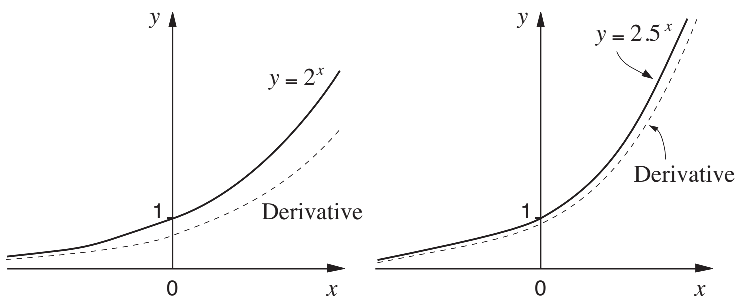
x	0	0.5	1.0	1.5	2
gradient of 2^x	0.70	0.98	1.39	1.96	2.77
gradient of 3^x	1.10	1.90	3.29	5.71	9.89



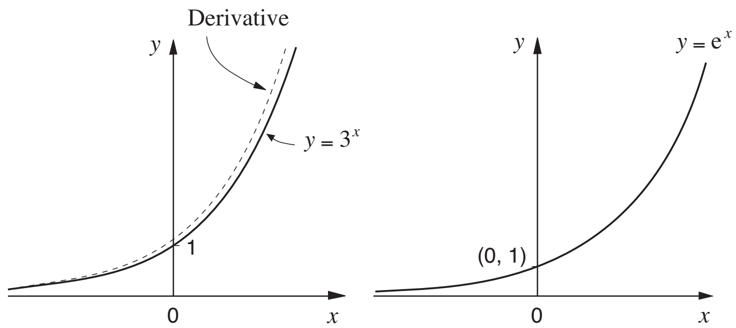
It is noted that the **gradient** curve of $y = 3^x$ is similar to $y = 3^x$ but with slightly larger values for $x > 0$ whilst the **gradient** curve of $y = 2x$ again is similar to $y = 2$ but with slightly lower values.

If you have access to a computer or calculator that is capable of showing the derivative, that is, the gradient, of a function, then you can find an exponential function whose derivative exactly fits over its own graph by considering $y = a^x$ with a in the range $2.5 < a < 2.9$.

The derivative of 2^x is always less than the value of the function itself, as is the derivative of 2.5^x, although it is a closer fit to the function than that of 2^x.



The derivative of 3^x has a greater value than the function. This suggests that there is an exponential function, with a base between 2.5 and 3, which has its derivative the same as itself.



Such a function would therefore be its own derivative. The base required for this to happen is denoted by the letter 'e'.

Unfortunately, its value cannot be given exactly - like π and $\sqrt{2}$ it is irrational, and so it cannot be expressed exactly as a fraction or decimal. To five decimal places, it is 2.71828.

The function $f(x) = e^x$ is often referred to as the **exponential function**. It is unique in mathematics, in that it is its own derivative. This property makes it extremely important in many branches of the subject.

To summarise, we write this as

$$y = e^x \quad \Rightarrow \quad \frac{dy}{dx} = e^x$$

when the derivative, $\frac{dy}{dx}$, is the gradient of the functions.

4

Worked Example 2

(a) Use a graphic calculator, computer or spreadsheet to make sketches of these graphs.

(i) $y = e^x$ (ii) $y = e^{(x+1)}$ (iii) $y = e^{(x-2)}$

(iv) $y = e^x + 1$ (v) $y = e^{-x}$

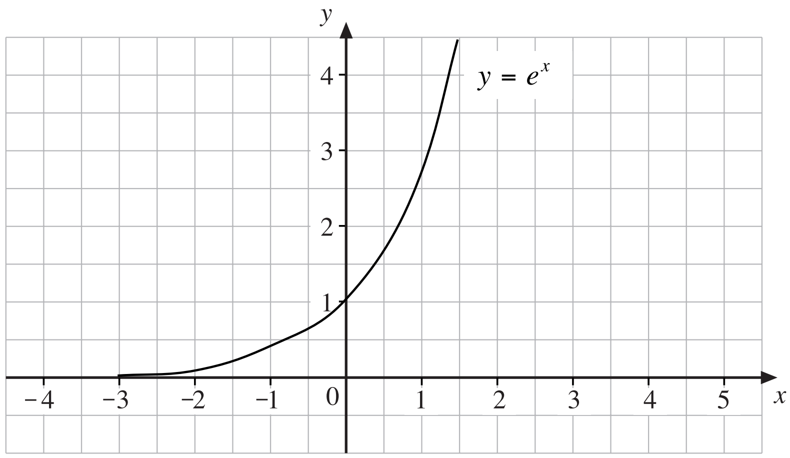
(Note that calculators or computers may use the expression $y = \exp(x)$ for $y = e^x$.)

(b) Compare each of the sketches with the graph $y = e^x$, and state the relationship between each graph and that of $y = e^x$.

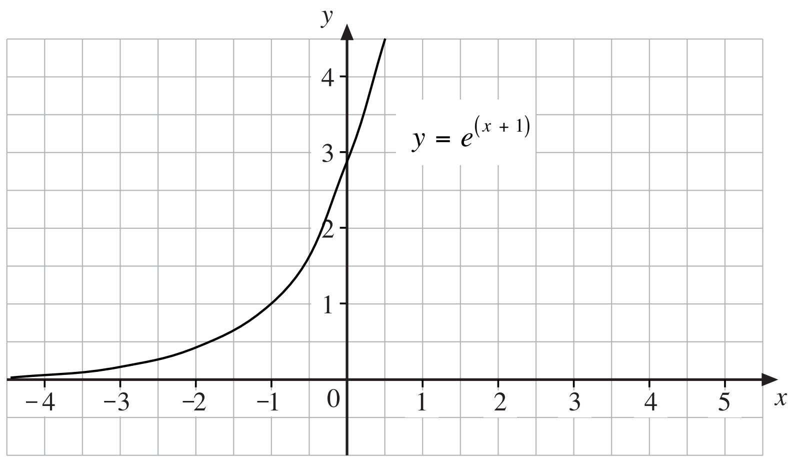
(c) Use the fact that the derivative of e^x is e^x, to work out the derivatives of each of the other functions.

Solution

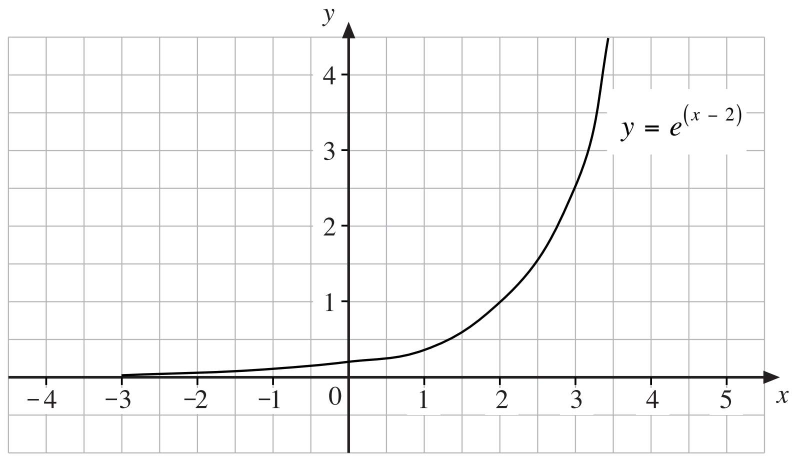
(a) (i)



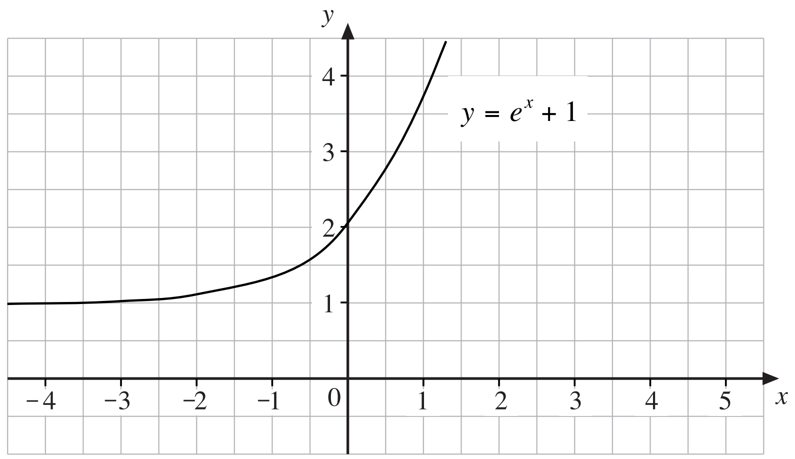
(ii)



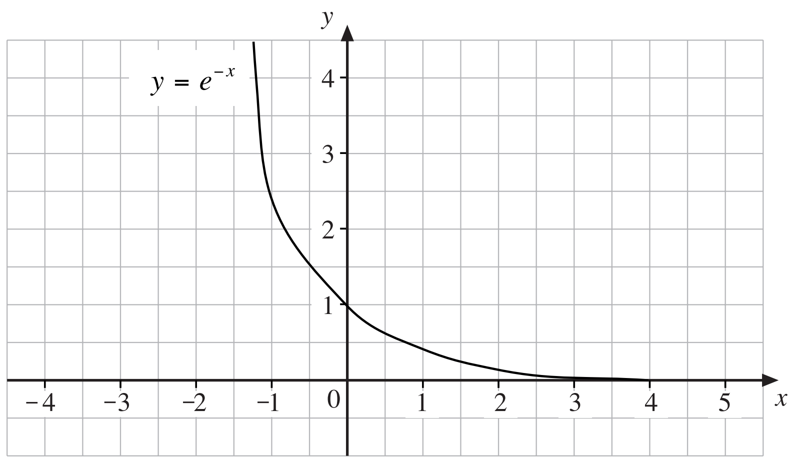
(iii)



(iv)



(v)



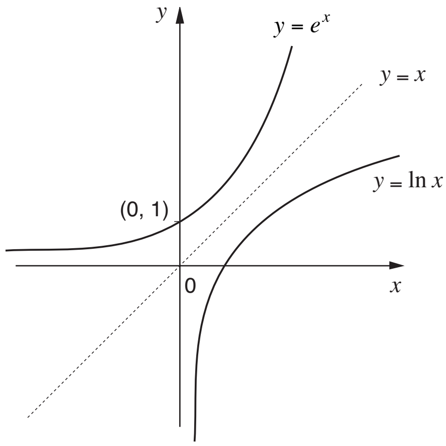
(b) (ii) Translation of $\begin{pmatrix}-1\\0\end{pmatrix}$ (iii) Translation of $\begin{pmatrix}2\\0\end{pmatrix}$

(iv) Translation of $\begin{pmatrix}0\\1\end{pmatrix}$ (v) Reflection in y axis

(c) (ii) e^{x+1} (iii) e^{x+2} (iv) e^{x} (v) $-e^{-x}$

The function $f(x) = e^x$ is a mapping from the set of real numbers, $\mathbb{R}$, to the positive real numbers. Its graph shows that it is a one to one function. This means that $f(x) = e^x$ has an inverse function. The graph of this inverse function is a reflection in the line $y = x$ of the graph of $y = e^x$.

The graph below shows e^x and its inverse function, which is usually written as $\ln(x)$. This function is read as 'the natural (or Naperian) logarithm of x' or 'the logarithm to base e of x'. (*John Napier,* 1550-1617, was a Scottish mathematician who originated the concept of logarithms.)



The figure shows that $\ln(x)$ is **not** defined for negative values of x (or zero), as there is no graph to the left of the y axis for $\ln(x)$. So $\ln(-2)$, for instance, does not exist. The range of $\ln(x)$, however, is the full set of real numbers.

Worked Example 3

Find x if $e^x = 100$. Give your answer to two d.p.

Solution

Since $e^x = 100$, and $y = \ln x$ is the inverse function of e^x,

$$x = \ln 100$$

Using a calculator to find ln (100), gives $x = 4.61$ to 2 d.p.

To summarise, for $a > 0$,

$$\boxed{e^x = a \Rightarrow x = \ln a}$$

Note that the brackets round 'a' in $\ln a$ have been omitted and will be in future except where it might cause confusion.

4

Worked Example 4

Solve, to 3 s.f. the equation $3e^{2x-1} = 5$,

Solution

Since $3e^{2x-1} = 5$, then

$$e^{2x-1} = \tfrac{5}{3}$$

Since e^x and $\ln x$ are inverse functions,

$$2x - 1 = \ln \tfrac{5}{3}$$

$$\Rightarrow \quad 2x = 1 + \ln \tfrac{5}{3}$$

$$\Rightarrow \quad x = \tfrac{1}{2}\left(1 + \ln \tfrac{5}{3}\right) = 0.755 \text{ to 3 s.f.}$$

Exercises

1. Solve $e^x = 5$ to 2 d.p.
2. Solve $e^x = \frac{1}{2}$ to 2 d.p.
3. Solve $4e^x = 3$ to 3 s.f.
4. Solve $e^{2x} = 1$ to 2 d.p.
5. Solve $3e^{\frac{1}{2}x} = 4$ to 3 s.f.
6. Solve $e^{-x} = 1.5$ to 2 d.p.
7. Solve $4e^{3x-2} = 16$ to 1 d.p.
8. Solve $7e^{3-x} = 2$ to 3 s.f.
9. Solve $e^x \times e^x = 3$ to 2 d.p.
10. Solve $e^{2x} = 4e^x$ to 3 s.f.

5 Solving Exponential Equations

Earlier in this unit, you have produced exponential functions as models, and then used graphs to estimate the solution to a problem. The logarithmic function allows you to calculate rather than estimate these solutions as is shown in the example below.

Worked Example 1

A bacteria colony doubles in number every minute, from a starting population of one. The population model is $P = 2^m$, where P is the population and m the number of minutes since the colony was started. Find the time when the population first equals 1000.

Solution

The problem requires a solution to the equation

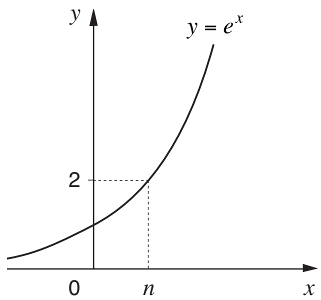
$$P = 1000$$

or $$2^m = 1000$$

Taking log of each side of the equation

$$\ln 2^m = \ln 1000 \qquad (1)$$

Now, 2 is a positive real number, so there is some number, call it n, such that $e^n = 2$ (see figure below). Then $n = \ln(2) \approx 0.693$.



So $2^m = (e^n)^m$ replacing 2 by e^n and since $(e^n)^m = e^{nm}$, using the properties of indices, 2^m in (1) above can be replaced by e^{nm}, where $n = \ln(2)$.

This gives

$$\ln e^{nm} = \ln 1000$$

But $\ln x$ is the inverse of e^x, so $\ln e^{nm} = nm$. Hence

$$mn = \ln 1000$$

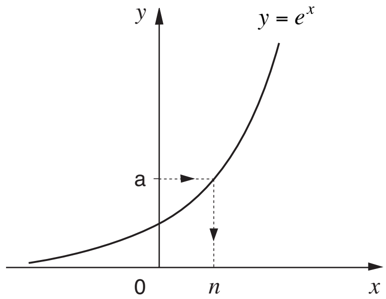
Therefore $m = \dfrac{\ln 1000}{n} = \dfrac{\ln 1000}{\ln 2} = 9.97$ minutes.

Hence $m = 9$ minutes, 58 seconds to the nearest second.

The example illustrates how a general method for solving exponential equations works. This process can be made quicker by using the results developed below.

Consider the function a^x, where $a > 0$.

As a is a number greater than zero, there is a real number, n, such that $e^n = a$ (see figure below. This means that $n = \ln a$, since $\ln x$ is the inverse function for the exponential function e^x.



So $a^x = (e^n)^x$ replacing a by e^n.

That is, $a^x = e^{xn}$ using laws of indices, and taking logarithms of both sides gives the equation

$$\ln a^x = \ln e^{xn} = xn$$

But $n = \ln a$ so

$$\boxed{\ln a^x = x \ln a}$$

This result is a great help in solving a wide variety of exponential equations.

Worked Example 2

Solve $3^{2x-1} = 5^x$, giving your answer to 2 d.p.

Solution

Since

$$3^{2x-1} = 5^x$$

$$\Rightarrow \quad \ln(3^{2x-1}) = \ln 5^x$$

$$\Rightarrow \quad (2x-1)\ln 3 = x \ln 5$$

$$\Rightarrow \quad 2x \ln 3 - \ln 3 = x \ln 5$$

$$\Rightarrow \quad 2x \ln 3 - x \ln 5 - \ln 3 = 0$$

$$\Rightarrow \quad 2x \ln 3 - x \ln 5 = \ln 3$$

$$\Rightarrow \quad x(2\ln 3 - \ln 5) = \ln 3$$

$$\Rightarrow \quad x = \frac{\ln 3}{2\ln 3 - \ln 5} = 1.87 \text{ to 2 d.p.}$$

Worked Example 3

A sample of wood has ^{14}C radioactivity of 6 becquerels per gram. New wood has ^{14}C radioactivity of 6.68 becquerels per gram of Carbon 14. The half life of ^{14}C is 5730 years;. Form a model based on the work in Section 5.4 for the ^{14}C radiation in wood, of the form $R = ba^t$, where R is the radioactivity, b and a are constants, and t is the time in years since the sample was formed.

Use your equation to find to the nearest year when $R = 6$ becquerels per gram of carbon.

Solution

Assume a model of the form $R = ba^t$.

Since the half life of ^{14}C is 5730 years, we have

$$\text{at } t = 0, \qquad R_0 = ba^0 = b$$

$$\text{at } t = 5730, \qquad \frac{R_0}{2} = ba^{5730} = R_0 a^{5730}$$

Hence a is given by $\frac{1}{2} = a^{5730}$ or $5730 \ln a = \ln(0.5)$.

For the wood test at time t after sample was formed

$$R_0 = 6.68 \text{ and } R(t) = 6$$

Thus

$$6 = 6.68a^t$$

$$\frac{6}{6.68} = a^t$$

giving

$$t \ln a = \ln\left(\frac{6}{6.68}\right)$$

So
$$t = \frac{\ln\left(\frac{6}{6.68}\right)}{\ln a}$$

$$= \ln\left(\frac{6}{6.68}\right) \times \frac{5730}{\ln(0.5)}$$

$$\approx 887.5 \text{ years}$$

So the sample was formed about 890 years ago (to the nearest decade).

5

Exercises

1. Solve $2^x = 5$ to 2 d.p.

2. Solve $3^{\frac{1}{2}x} = 1$ to 2 d.p.

3. Solve $4 \times 2^x = 3$ to 3 s.f.

4. Solve $3^x = 5$ to 2 d.p.

5. Solve $2^{-x} = 6$ to 3 s.f.

6. Solve $3^{2x} = 4$ to 2 d.p.

7. Solve $5^{x-1} = 3$ to 3 s.f.

8. Solve $2^{2x+1} = 4$ to 2 d.p.

9. Solve $5^{x-1} = e^{2x}$ to 1 d.p.

10. Solve $6^{2x+1} = 3^{-x}$ to 2 d.p.

6 Properties of Logarithms

As well as obeying the rule

$$\ln(a^x) = x \ln a$$

logarithms also obey, for any real numbers a, b,

$$\ln(ab) = \ln a + \ln b \qquad (1)$$

and

$$\ln\left(\frac{a}{b}\right) = \ln a - \ln b \qquad (2)$$

To prove the first result, (1), note that a and b can be written in the form

$$a = e^m, b = e^n$$

for some real numbers m and n. Then

$$\begin{aligned} \ln(ab) &= \ln(e^m e^n) \\ &= \ln(e^{m+n}) \\ &= m + n \end{aligned}$$

Since $\ln x$ is the inverse function of e^x,

$$e^x \ln a = \ln(e^m) = m$$

$$\ln b = \ln(e^n) = n$$

so that

$$\ln(ab) = \ln a + \ln b$$

You will see how useful these results are in the following applications.

Before the theory of gravitation was developed by *Sir Isaac Newton*, the best laws available to describe planetary motion were those formulated by *Johann Kepler*, a German astronomer. His laws were based on his own meticulous observations, and were used later as a 'benchmark test' for Newton's own theory. The next example investigates Kepler's third law.

Worked Example 1

This table shows how the average radius of a planet's orbit around the Sun, R, is related to the period of that orbit in years, T. (The orbits are elliptical, not circular, so an average radius is used here). Only the planets known to Kepler are included.

Planet	*Radius, R* (millions of km)	*Period, T* (years)
Mercury	57.9	0.24
Venus	108.2	0.62
Earth	149.6	1
Mars	227.9	1.88
Jupiter	778.3	11.86
Saturn	1427.0	29.46

Solution

Assuming that T and R are linked by a model of the form $T = aR^b$, we can use the data to find estimates for the constants a and b, together with the properties of logarithms.

Assuming a power law of the form

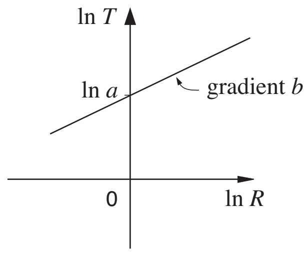
$$T = aR^b,$$

taking logs of each side gives

$$\begin{aligned} \ln T &= \ln(a\mathrm{R}^b) \\ &= \ln a + \ln\left(R^b\right) \quad \text{(using equation (1))} \\ &= \ln a + b \ln \mathrm{R} \end{aligned}$$

This equation resembles a straight line equation $y = mx + c$ with y replaced by $\ln T$ and x by $\ln R$. So a graph of $\ln T$ against $\ln R$ should give a straight line and the constants a and b can be estimated from the graph. The constant b will be the **gradient** of the line, and ln a will be the **intercept** on the vertical axis.

Unfortunately, for this data $\ln_a$ is negative and difficult to estimate from your graph. Having estimated b (the gradient) you can obtain an estimate of a by considering the point $\ln T = 0$.

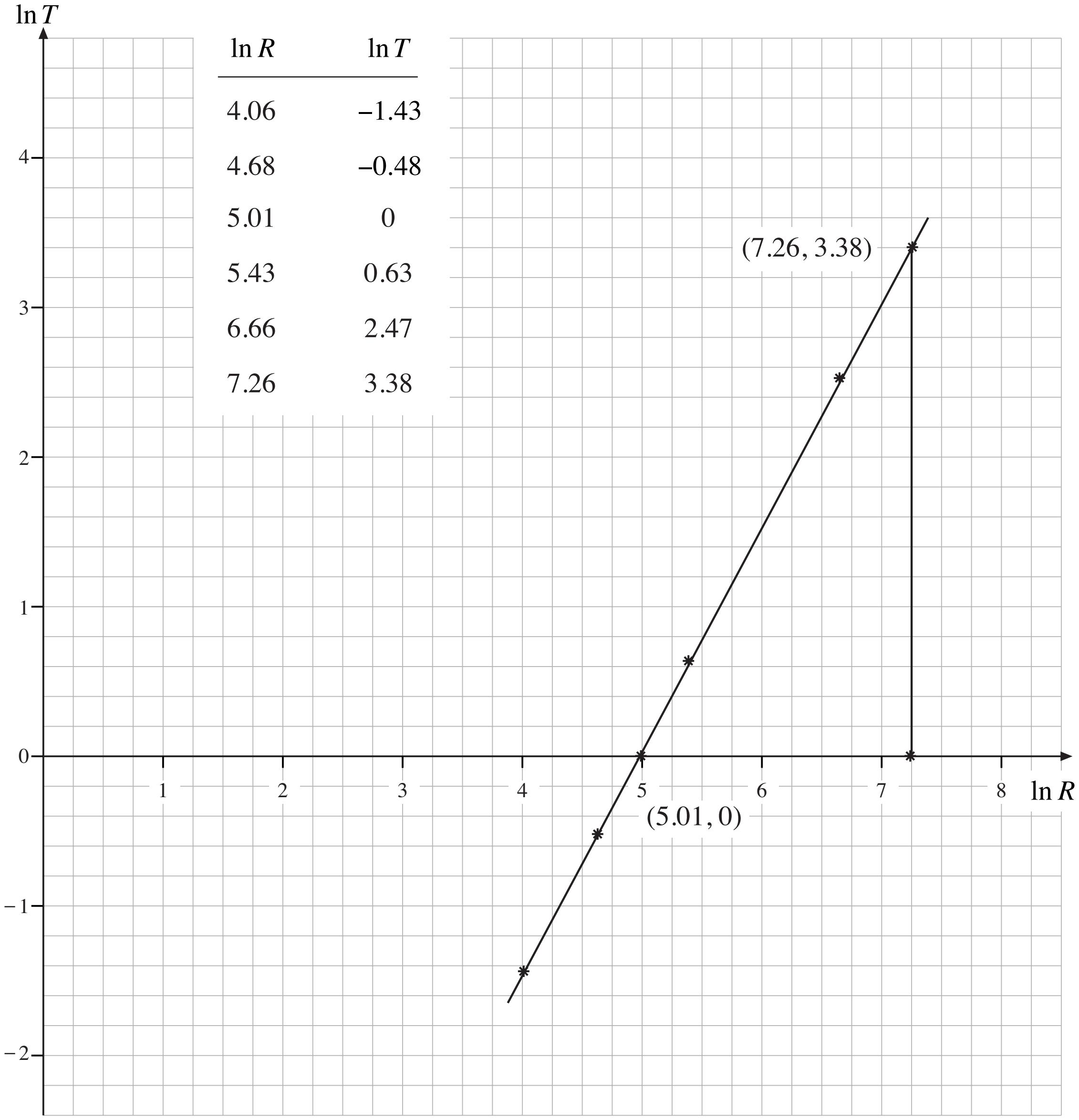


Worked Example 2

For the data above, plot a graph of $\ln T$ against $\ln R$, and use it to estimate the values of the constants a and b.

Solution

$\ln R$	$\ln T$
4.06	–1.43
4.68	–0.48
5.01	0
5.43	0.63
6.66	2.47
7.26	3.38



$$\text{Gradient} = \frac{3.38 - 0}{7.26 - 5.01} = \frac{3.38}{2.25} = 1.502$$

$$\approx 1.5$$

To find an estimate of a, note that when $\ln T = 0$,

$$\ln a + b \ln R = 0$$

i.e. $$\ln a \approx -1.5 \ln(5.01) \approx 2.42$$

Hence $$T = 2.42 \times R^{\frac{3}{2}}$$

The note produced by a musical instrument is directly related to its frequency (the number of times the air is caused to vibrate every second). The higher the frequency, the higher the note. In order to set the frets on a guitar in the correct place, the maker must know how the length of a string affects the frequency of the note it produces.

Worked Example 3

This relationship between length, l (cm), and frequency, f (hz), can be found experimentally. The table shows some data collected by experiment for a particular type of string.

Length l (cm)	50	60	70	80	90	100
Frequency f (hz)	410	330	275	255	225	195

The relationship is assumed to be of the form $f = al^b$ where a and b are constant.

Use logarithms to 'linearise' the relationship, as described previously. Plot $\ln f$ on a vertical axis and $\ln l$ on the horizontal, and draw a line of best fit. Find the gradient and intercept with the vertical axis of this line, and so determine the values of a and b.

Solution

If $f = al^b$ for constants a and b,

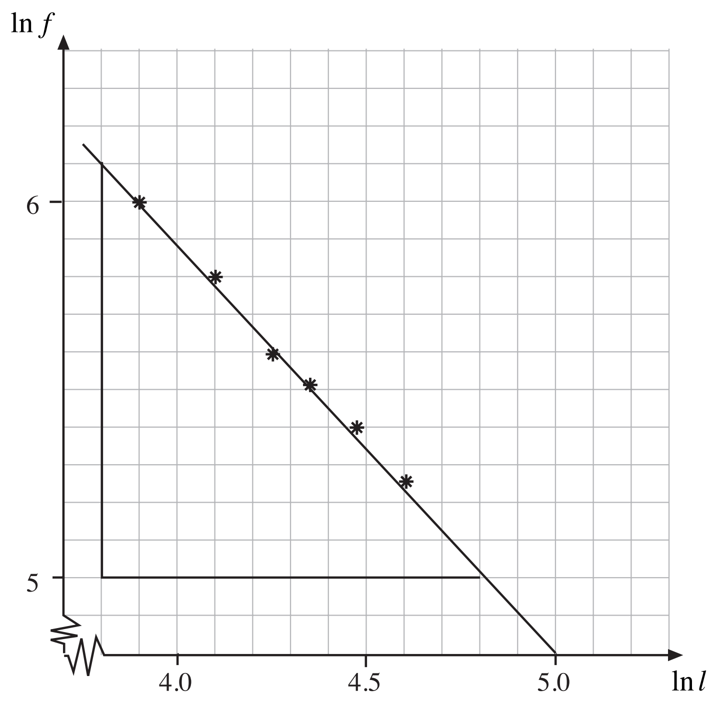
then

$$\ln f = \ln a + b \ln l$$

To estimate values of a and b, we graph $\ln f$ against $\ln l$.

$\ln f$	$\ln l$
6.02	3.91
5.80	4.09
5.62	4.25
5.54	4.38
5.42	4.50
5.28	4.61

6



Gradient $\approx \dfrac{5 - 6.1}{4.75 - 3.8} \approx -1.16 = b$

You can best find an estimate for a by substituting values, for example,

$$\ln f = 6.02 \text{ when } \ln l = 3.91$$

This gives $\quad 6.02 = \ln a - 1.16 \times 3.91$

i.e. $\quad \ln a = 27.30$

$$a \approx 7.2$$

Hence you can deduce that

$$f \approx 7.2 l^{-1.16}$$

The frequencies produced are also affected by the tension in the string and so, even with frets correctly placed, the guitarist must still 'tune' the instrument by changing the tensions in the strings.

'Middle C' has a frequency of 264 Hz.

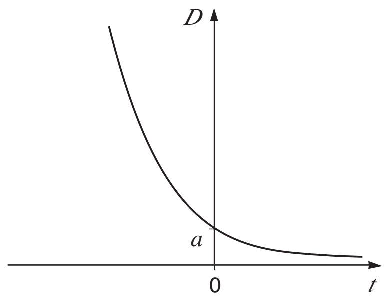
The final application in this section is based on the method used by forensic scientists to estimate the time of death of a body.

When a person dies, the body's temperature begins to cool. The temperature of the body at any time after death is governed by Newton's Law of Cooling, which applies to any cooling object:

$$D = ae^{-kt}$$

D is the temperature difference between the cooling object and its surrounding, a and k are constants, and t is the time since the object started to cool. The values of a and k depend on the size, shape and composition of the object and the initial temperature difference.

If D is plotted against the time, t, the graph will be similar to the curve shown opposite.



To find out the equation of the curve which applies to a dead body, the values of a and k must be found. This will require two readings of its temperature.

Worked Example 4

The police arrive at the scene of a murder at 8 a.m.

On arrival, the temperature of the body and its surroundings are measured at $34°$C and $17°$C respectively. This was taken to be the moment when the time, t, was equal to zero.

At 9 a.m. when $t = 1$, the body temperature was measured as $33°$C and the room temperature still as $17°$C.

Estimate the time of death.

Solution

The two sets of data are

$$D = 34 - 17 = 17 \text{ at } t = 0$$

$$D = 33 - 17 = 16 \text{ at } t = 1$$

Substituting in the governing equation

$$D = ae^{-kt}$$

gives

$$17 = ae^{-k.0} = ae^0 = a$$

(since $e^0 = 1$); and

$$16 = ae^{-k.1} = ae^{-k} = 17e^{-k}$$

Therefore

$$e^{-k} = \frac{16}{17}$$

and taking 'logs',

$$-k = \ln\left(\frac{16}{17}\right) = -0.0606 \Rightarrow k = 0.0606$$

Hence

$$D = 17e^{-0.0606t} \qquad (3)$$

Now normal body temperature is given by $36.9°C$, so the corresponding value of D is given by

$$D = 36.9 - 17 = 19.9$$

Substituting this value of D into equation (3) and solving for t will give you the estimated time of death; this gives

$$19.9 = 17e^{-0.0606t}$$

$$\Rightarrow \quad e^{-0.0606t} = \frac{19.9}{17}$$

$$\Rightarrow \quad -0.0606t = \ln\left(\frac{19.9}{17}\right)$$

$$\Rightarrow \quad t = -\frac{1}{0.0606}\ln\left(\frac{19.9}{17}\right)$$

$$= -2.599 \text{ hours}$$

$$\approx -(2 \text{ hours } 36 \text{ minutes}).$$

So the estimated time of death is estimated at 5.24 am, or about 5.30 am.

Exercises

1. A body is found at 11.30 pm. The body temperature at midnight is found to be $33°C$ and at 2.00 am it is $31.5°C$. Assuming the surroundings are at a constant temperature of $20°C$, estimate the time of death.

2. A physicist conducts an experiment to discover the half life of an element. The radioactivity at one moment from a sample of the element is measured as 30 becquerels. One hour later the radioactivity is just 28 becquerels. Assuming that the radioactivity is governed by a formula of the form

$$R = a \times 2^{-kt}$$

where R is the radioactivity in becquerels per gram, t the time in hours, and a and k are constants, find the values of a and k, and hence determine the half life in hours.