CONGRUENCE AND SIMILARITY

Text

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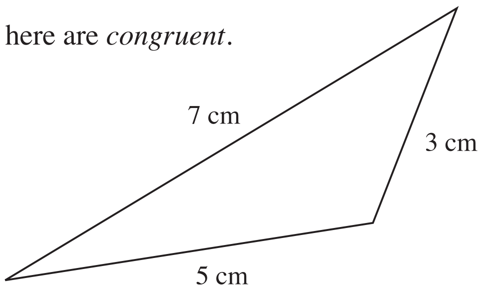
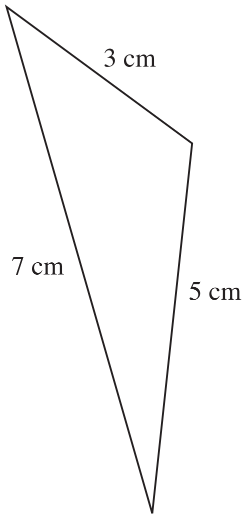
Congruence and Similarity

1 Congruence

Two shapes are said to be *congruent* if they are the same shape and size: that is, the corresponding sides of both shapes are the same length and corresponding angles are the same.

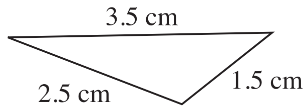
The two triangles shown here are *congruent*.





Shapes which are of different sizes but which have the same shape are said to be *similar*.

The triangle below is *similar* to the triangles above but because it is a different size it is *not congruent* to the triangles above.

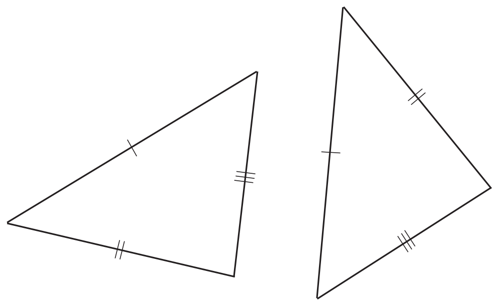


There are four tests for congruence which are outlined below.

TEST 1 (Side, Side, Side)

If all three sides of one triangle are the same as the lengths of the sides of the second triangle, then the two triangles are congruent.

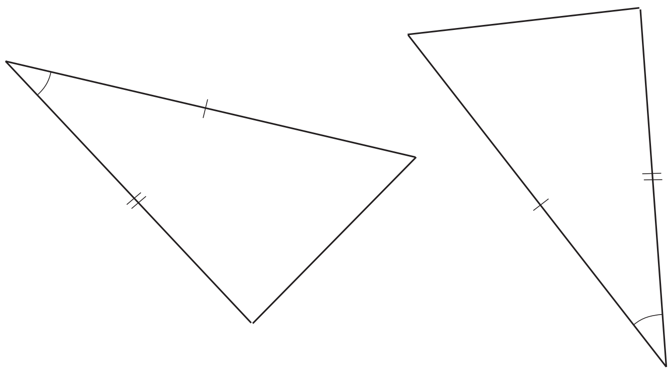
This test is referred to as *SSS*.



TEST 2 (Side, Angle, Side)

If two sides of one triangle are the same length as two sides of the other triangle and the angle *between* these two sides is the same in both triangles, then the triangles are congruent.

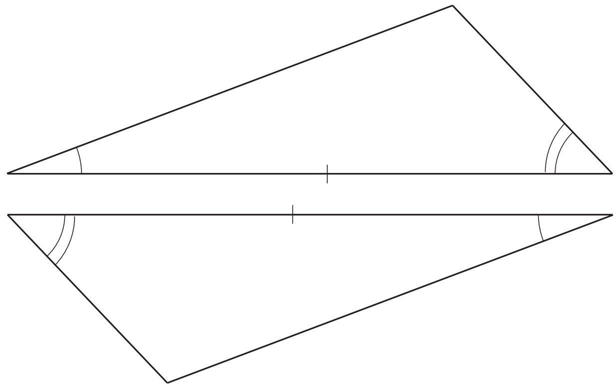
This test is referred to as *SAS*.



TEST 3 (Angle, Angle, Side)

If two angles and the length of one corresponding side are the same in both triangles, then they are congruent.

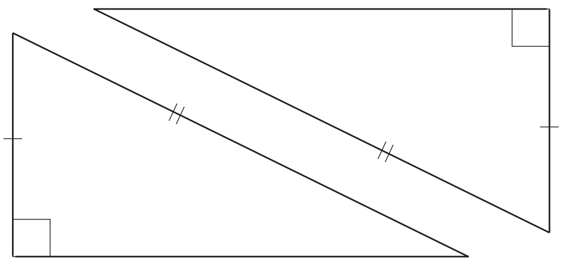
This test is referred to as *AAS*.



TEST 4 (Right angle, Hypotenuse, Side)

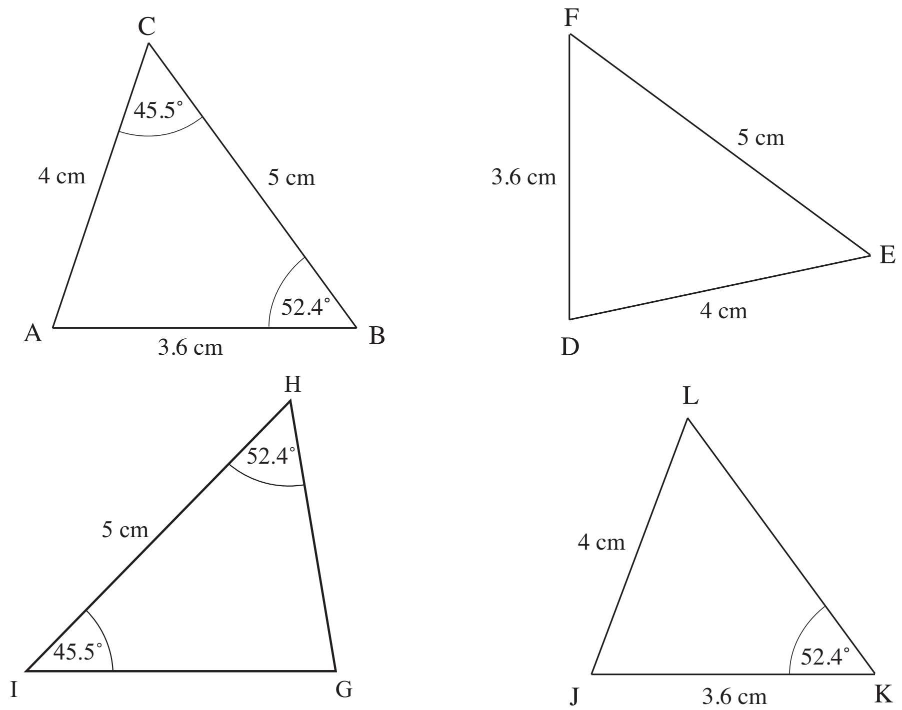
If both triangles contain a right angle, have hypotenuses of the same length and one other side of the same length, then they are congruent.

This test is referred to as *RHS*.



Worked Example 1

Which of the triangles below are congruent to the triangle ABC, and why?



Solution

Consider first the triangle DEF:

AB = DF
BC = EF
AC = DE

As the sides lengths are the same in both triangles the triangles are congruent. *(SSS)*

Consider the triangle GHI:

$$BC = HI$$
$$A\hat{B}C = G\hat{H}I$$
$$A\hat{C}B = G\hat{I}H$$

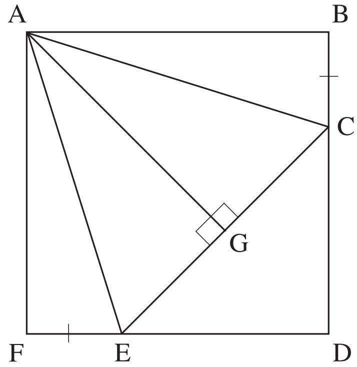
As the triangles have one side and two angles the same, they are congruent. *(AAS)*

Consider the triangle JKL: Two sides are known but the angle between them is unknown, so there is insufficient information to show that the triangles are congruent.

Worked Example 2

ABDF is a square and $BC = EF$.

Find the pairs of congruent triangles in the diagram.



Solution

Consider the triangles ABC and AFE:

$AB = AF$		(ABDF is a square.)
$BC = FE$		(This is given in the question.)
$A\hat{B}C = A\hat{F}E = 90°$		(They are corners of a square.)

The triangles ABC and AFE have two sides of the same length and also have the same angle between them, so these triangles are congruent. *(SAS)*

Consider the triangles ACG and AEG:

$AC = AE$		(Δ ABC and Δ AFE are congruent.)
$AG = AG$		(They are the same line.)
$E\hat{G}A = C\hat{G}A = 90°$		(This is given in the question.)

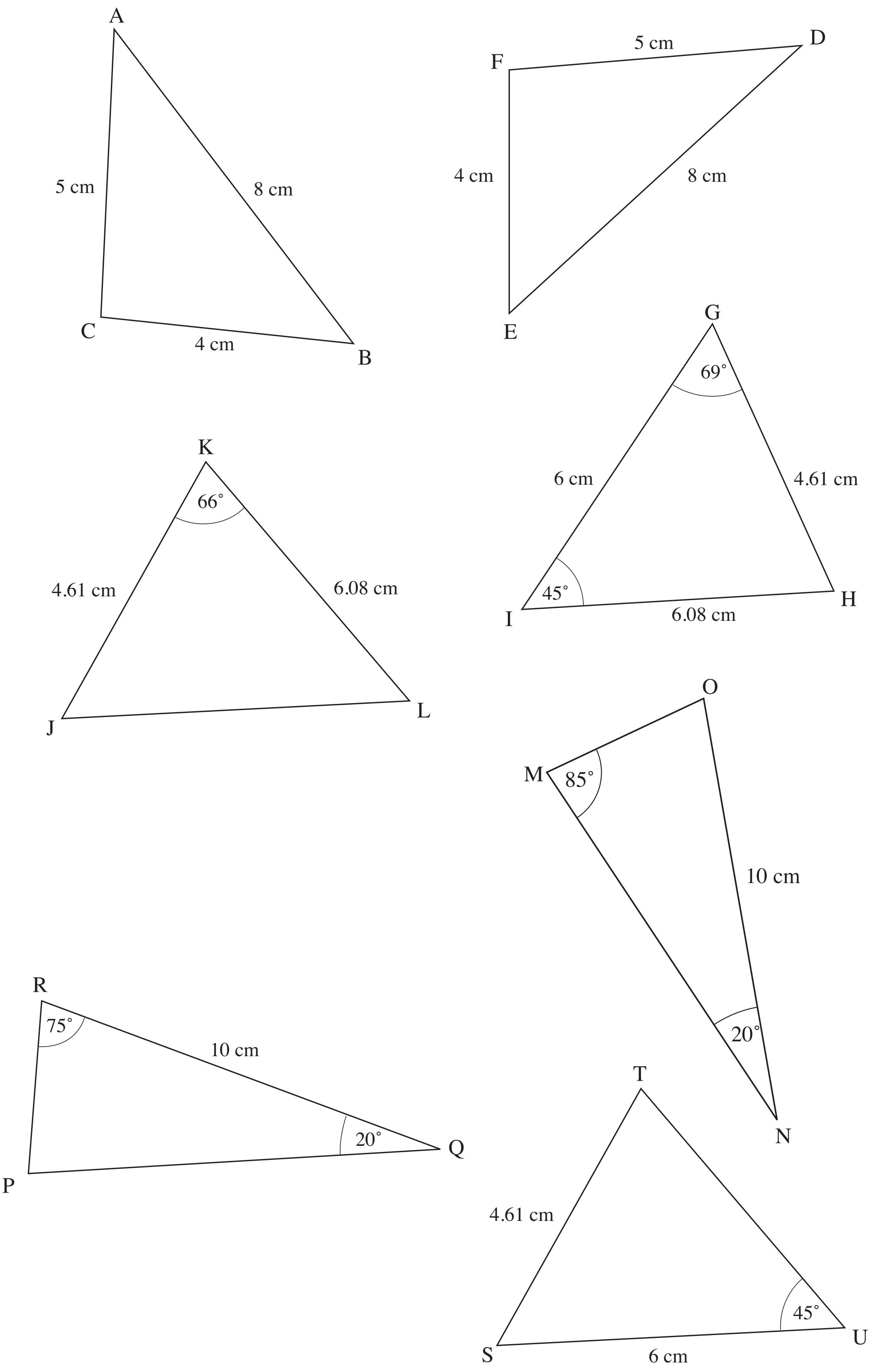
Both triangles contain right angles, have the same length hypotenuse and one other side of the same length. So the triangles are congruent. *(RHS)*

Investigation

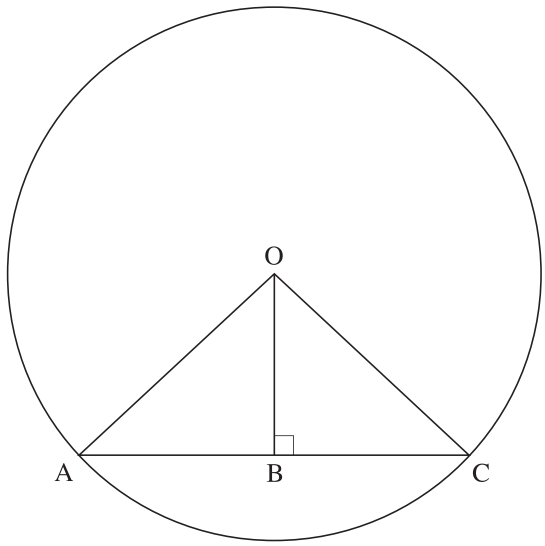
1. *How many straight lines can you draw to divide a square into two congruent parts?*
2. *How many lines can you draw to divide a rectangle into two congruent parts?*
3. *Can you draw two straight lines through a square to divide it into four congruent quadrilaterals which are **not** parallelograms?*

Exercises

1. Identify the triangles below which are congruent and give the reasons why.



2. If O is the centre of the circle, prove that the triangles OAB and OCB are congruent.



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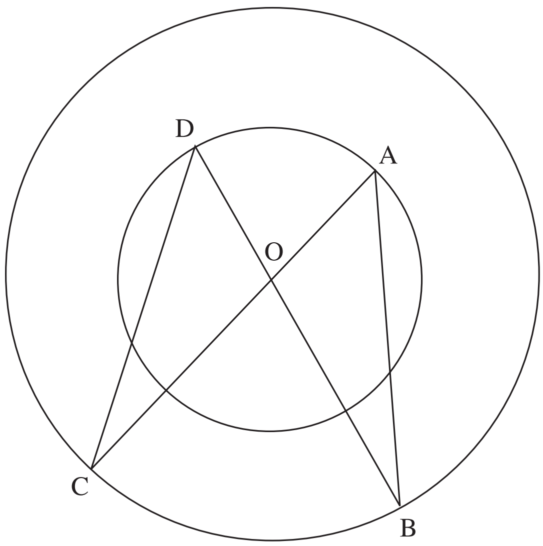
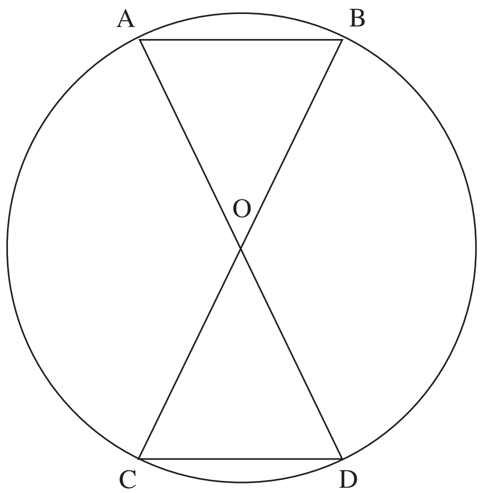
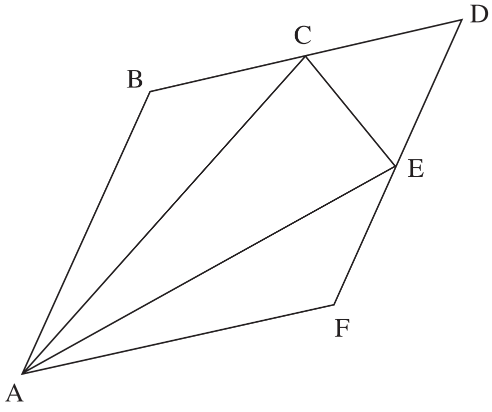


If O is the centre of both circles, prove that the triangles OAB and ODC are congruent.

4. If O is the centre of the circle, prove that the triangles OAB and OCD are congruent.



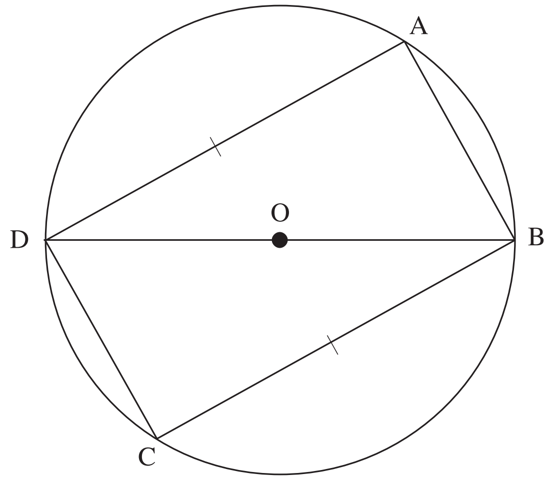
5.



When BC = EF, this rhombus contains two congruent triangles.

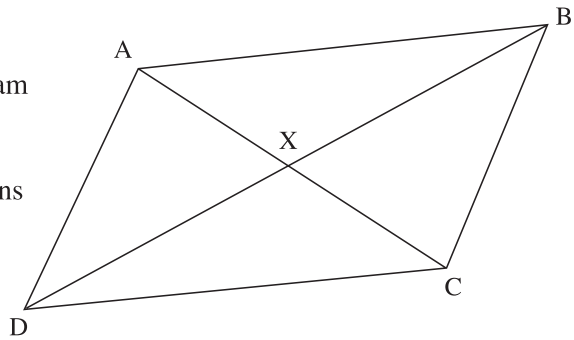
Identify the triangles and prove that they are congruent.

6. If O is the centre of the circle and $AD = BC$, show that ABD and CDB are congruent triangles.



7. Two triangles have sides of lengths 8 cm and 6 cm, and contain an angle of $30°$. Show that it is possible to draw 4 different triangles, none of which are congruent, using this information.

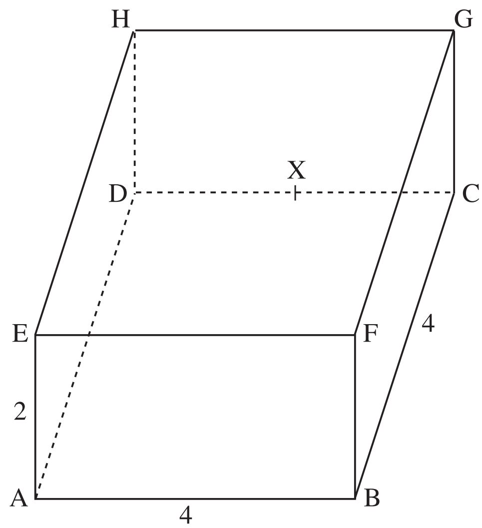
8. The diagram shows the parallelogram ABCD with diagonals which intersect at X. Prove that the parallelogram contains 2 pairs of congruent triangles.



9. The diagram shows a cuboid. The mid-point of the side DC is X.

   (a) Show that triangles AHG and FAD are congruent.

   (b) Show that triangles ADX and BCG are congruent.



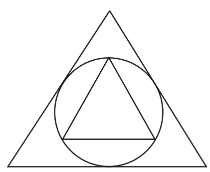
Investigation

1. In how many ways can you cut a square-based cake into two congruent parts?

2. How can you use eight straight lines of equal length to make a square and four congruent equilateral triangles?

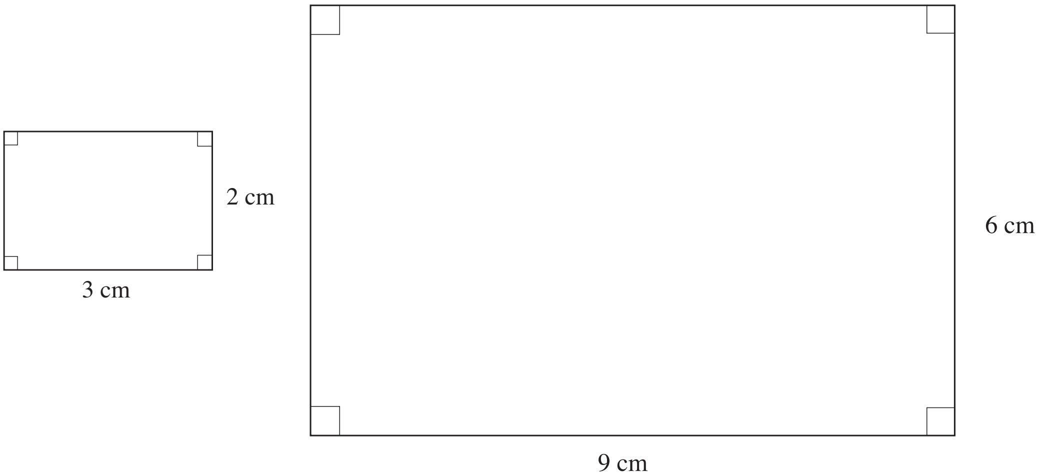
Challenge!

Study the diagram opposite and then find the ratio of the area of the large triangle to that of the small triangle.



2 Similarity

Similar shapes have the same shape but may be different sizes. The two rectangles shown below are similar – they have the same shape but one is smaller than the other.



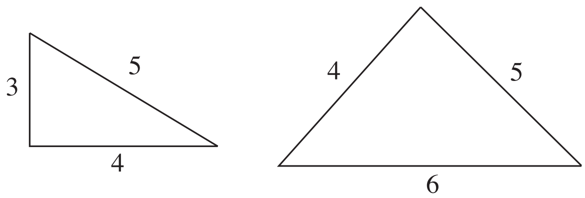
They are similar because they are both rectangles and the sides of the larger rectangle are three times longer than the sides of the smaller rectangle.

It is interesting to compare the area of the two rectangles. The area of the smaller rectangle is $6\ \text{cm}^2$ and the area of the larger rectangle is $54\ \text{cm}^2$, which is nine times $\left(3^2\right)$ greater.

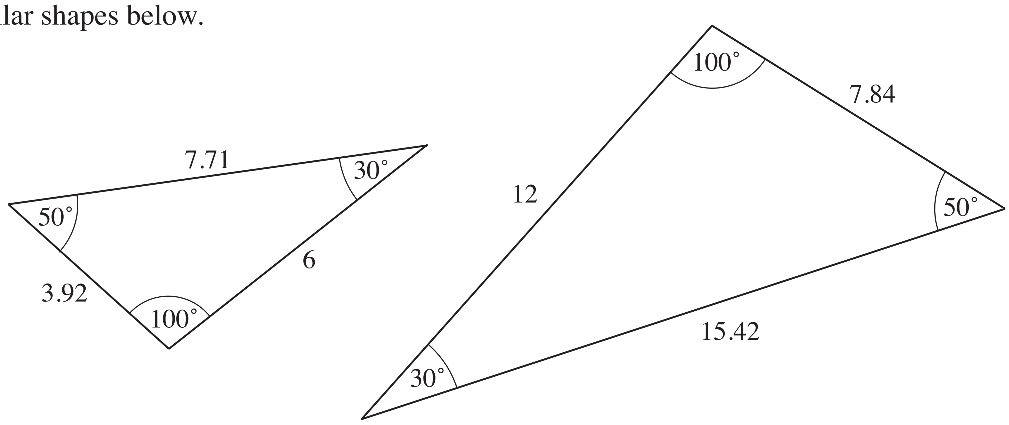
Note

In general, if the lengths of the sides of a shape are increased by a factor k, then the area is increased by a factor k^2.

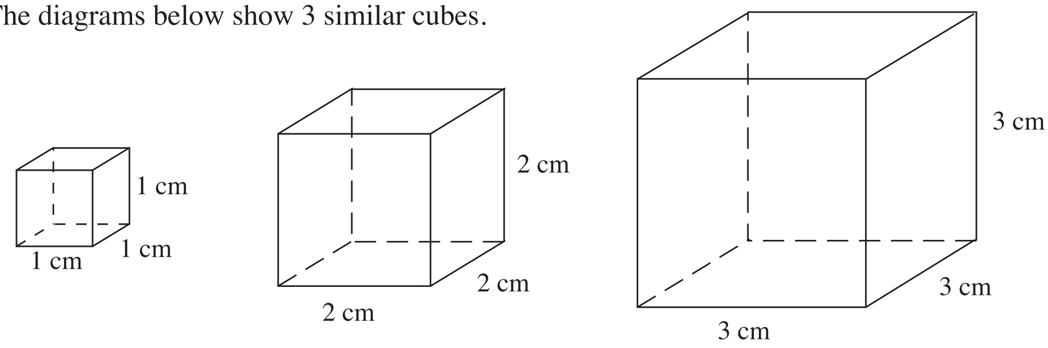
These two triangles are *not* similar. The sides lengths of the triangles are not in the same ratio and so the triangles are not similar.



For two triangles to be similar, they must have the same internal angles, as shown in the similar shapes below.



The diagrams below show 3 similar cubes.



Length of side	*Area of one face*	*Surface area*	*Volume*
1 cm	1 cm^2	6 cm^2	1 cm^3
2 cm	4 cm^2	24 cm^2	8 cm^3
3 cm	9 cm^2	54 cm^2	27 cm^3

The table gives the lengths of sides, area of one face, total surface area and volume.

Comparing the larger cube with the 1 cm cube we can note that:

For the 2 cm cube

The lengths are	2	times greater.
The areas are	$4 = 2^2$	times greater.
The volume is	$8 = 2^3$	times greater.

For the 3 cm cube

The lengths are	3	times greater.
The areas are	$9 = 3^2$	times greater.
The volume is	$27 = 3^3$	times greater.

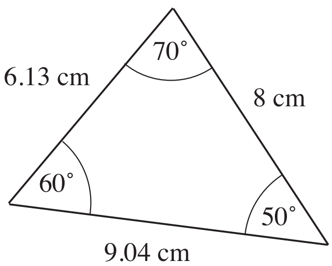
Note

If the lengths of a solid are increased by a factor, k, its surface area will increase by a factor k^2 and its volume will increase by a factor k^3.

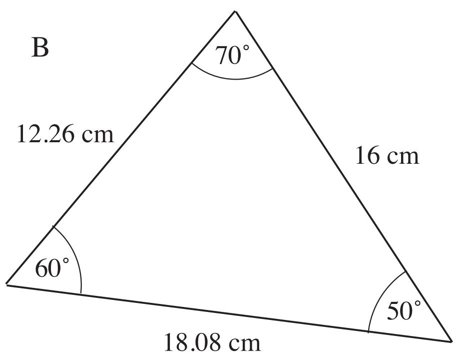
Worked Example 1

(a) Which of the triangles, A, B, C, D, shown below are similar?

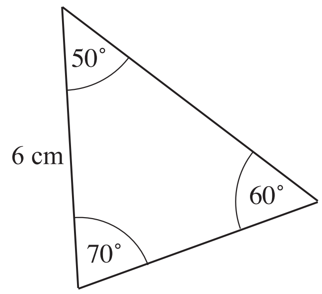
A



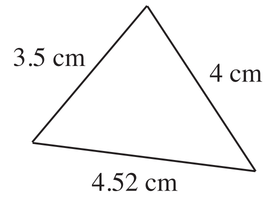
B



C



D



(b) How do the areas of the triangles which are similar compare?

Solution

(a) *First compare triangles A and B.*

Here all the lengths of the sides are twice the length of the sides of triangle A, so the two triangles are similar.

Then compare triangles A and C.

Here all the angles are the same in both triangles, so the triangles must be similar.

Finally, compare triangles A and D.

Note that $4 = \frac{1}{2} \times 8$ and $4.52 = \frac{1}{2} \times 9.04$, but $3.5 \neq \frac{1}{2} \times 6.13$.

So these triangles are *not* similar.

(b) The lengths of the sides of triangle B are 2 times greater than the lengths of the sides of triangle A, so the area will be $2^2 = 4$ times greater.

The side lengths of triangle C are $\frac{3}{4}$ of the side lengths of triangle A.

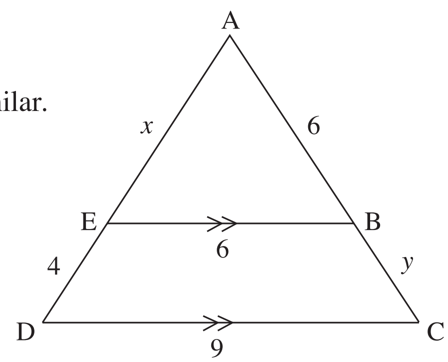
So the area will be $\left(\frac{3}{4}\right)^2 = \frac{9}{16}$ of the area of triangle A.

The ratio of the areas of triangles C : A : B can be written as:

$$\frac{9}{16} : 1 : 4$$

or $$9 : 16 : 64$$

Worked Example 2



(a) Explain why triangles ABE and ACD are similar.

(b) Find the lengths of x and y.

(c) Find the ratio of the area of ABE to BCDE.

Solution

(a) As the lines BE and CD are parallel,

$$A\hat{B}E = A\hat{C}D$$

and $$A\hat{E}B = A\hat{D}C$$

As the vertex A, is common to both triangles,

$$D\hat{A}C = E\hat{A}B$$

So the three angles are the same in both triangles and therefore they are similar.

(b) Comparing the sides BE and CD, the lengths in the larger triangle are 1.5 times the lengths in the smaller triangle. Alternatively, it can be stated that the ratio of the lengths is 2 : 3.

So the length AC will be 1.5 times the length AB.

$$AC = 1.5 \times 6$$
$$= 9$$
$$\text{So} \quad y = 3$$

In the same way, $AD = 1.5 \times AE$,

$$\text{so} \quad 4 + x = 1.5x$$
$$4 = 0.5x$$
$$x = 8$$

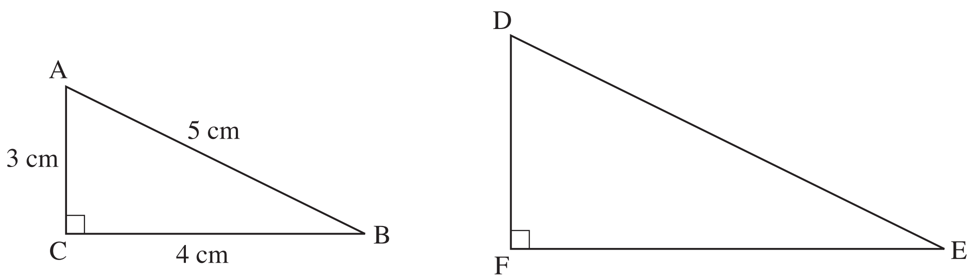
(c) As the lengths are increased by a factor of 1.5 or $\frac{3}{2}$ for the larger triangle, the areas will be increased by a factor of 1.5^2 or $\left(\frac{3}{2}\right)^2$. We can say that the ratio of the areas of the triangles is $1 : 2.25$ or $1 : \frac{9}{4}$ or $4 : 9$.

If the area of triangle ABE is $4k$, then the area of triangle ACD is $9k$ and hence the area of the quadrilateral BCDE is $5k$.

So the ratio of the area of ABE to BCDE is 4 : 5.

Worked Example 3

The diagrams show two similar triangles.



If the area of triangle DEF is 26.46 cm^2, find the lengths of its sides.

Solution

If the lengths of the sides of triangle DEF are a factor k greater than the lengths of the sides of triangle ABC, then its area will be a factor k^2 greater than the area of ABC.

$$\begin{aligned} \text{Area of ABC} &= \frac{1}{2} \times 4 \times 3 \\ &= 6 \text{ cm}^2. \end{aligned}$$

$$\begin{aligned} \text{So} \qquad 6 \times k^2 &= 26.46 \\ k^2 &= 4.41 \\ k &= \sqrt{4.41} \\ &= 2.1 \end{aligned}$$

So the lengths of the sides of triangle DEF will be 2.1 times greater than the lengths of the sides of triangle ABC.

$$\begin{aligned} \text{DE} &= 2.1 \times 5 \\ &= 10.5 \text{ cm} \end{aligned}$$

$$\begin{aligned} \text{DF} &= 2.1 \times 3 \\ &= 6.3 \text{ cm} \end{aligned}$$

$$\begin{aligned} \text{EF} &= 2.1 \times 4 \\ &= 8.2 \text{ cm} \end{aligned}$$

Worked Example 4

A can has a height of 10 cm and has a volume of 200 cm^3. A can with a similar shape has a height of 12 cm.

(a) Find the volume of the larger can.

(b) Find the height of a similar can with a volume of 675 cm^3.

Solution

(a) The lengths are increased by a factor of 1.2, so the volume will be increased by a factor of 1.2^3.

$$\begin{aligned} \text{Volume} &= 200 \times 1.2^3 \\ &= 345.6 \text{ cm}^3 \end{aligned}$$

(b) If the lengths are increased by a factor of k, then the volume will be increased by k^3.

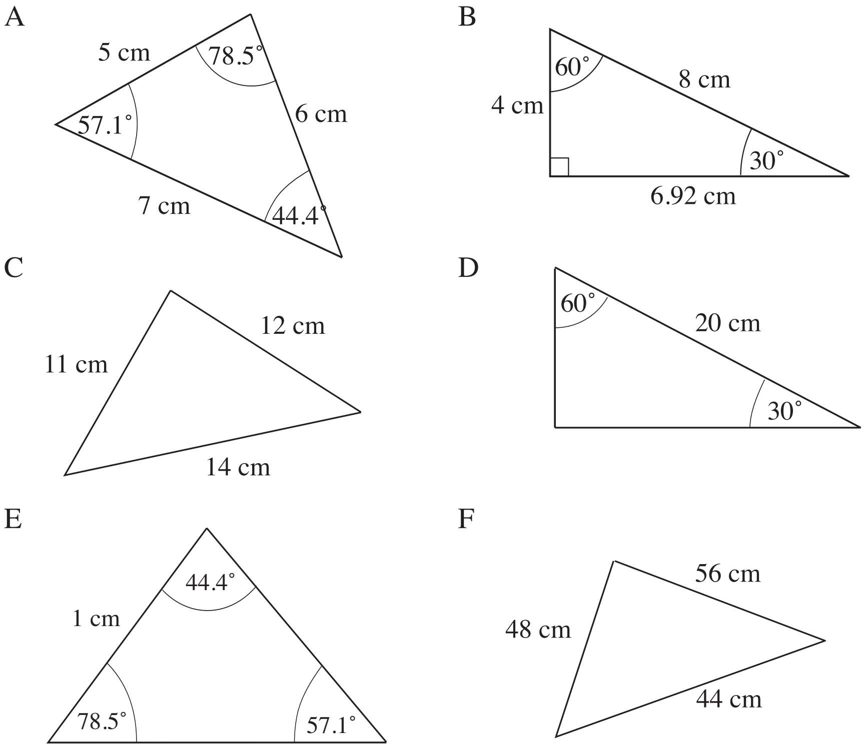
$$\begin{aligned} 675 &= 200 \times k^3 \\ k^3 &= 3.375 \\ k &= \sqrt[3]{3.375} \\ &= 1.5 \end{aligned}$$

So the height must be increased by a factor of 1.5, to give

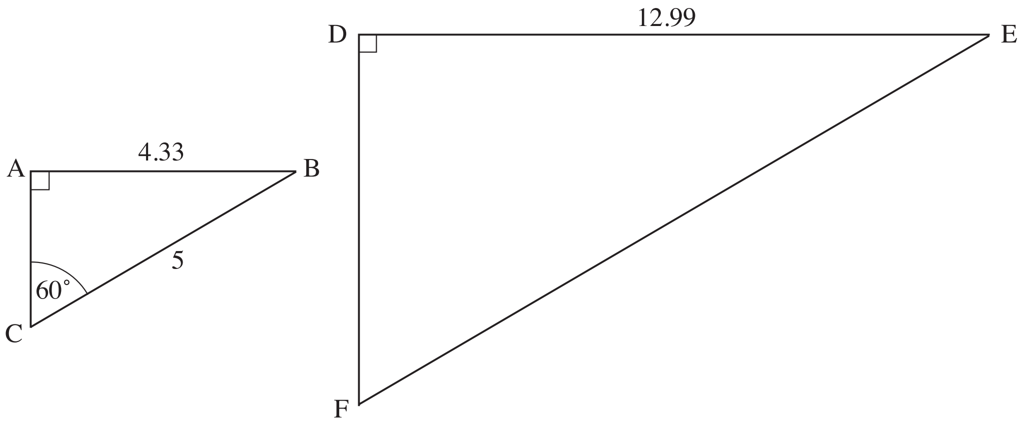
$$\text{height} = 1.5 \times 10$$
$$= 15 \text{ cm}$$

Exercises

1. Which of the triangles below are similar? Diagrams are not drawn to scale.



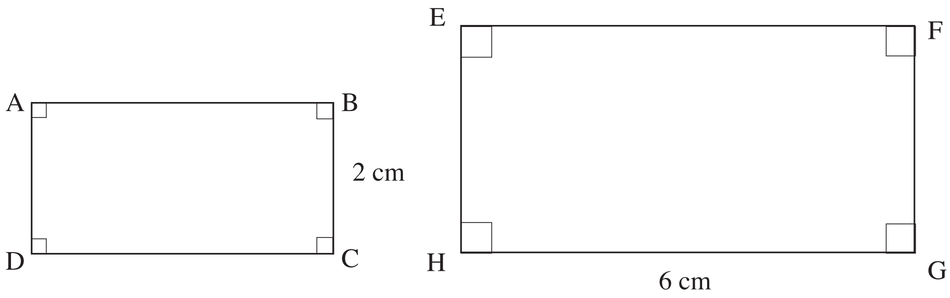
2. The diagram shows 2 similar triangles.



(a) Copy the triangles and label all the angles and the lengths of all the sides.

(b) How do the areas of the two triangles compare?
(Express your answer as a ratio.)

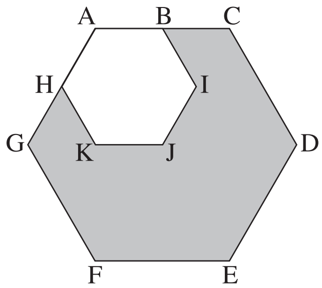
3. The diagram shows two similar rectangles, ABCD and EFGH.



Find the lengths of AB and EH if the ratio of the area of ABCD to the area of EFGH is:

(a) 1 : 4 (b) 4 : 9.

4. The diagram shows two regular hexagons and AB = BC.



(a) What is the ratio of the area of the smaller hexagon to the area of the larger hexagon?

(b) What is the ratio of the area of the smaller hexagon to that of the shaded area?

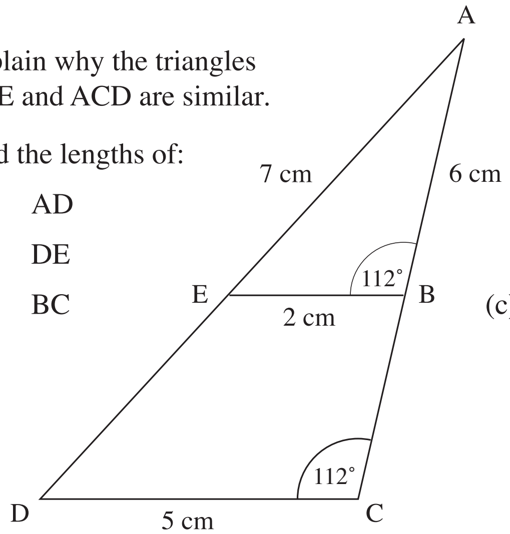
5. (a) Explain why the triangles ABE and ACD are similar.

(b) Find the lengths of:

(i) AD

(ii) DE

(iii) BC



(c) Find the ratio of the areas of:

(i) ABE to ACD

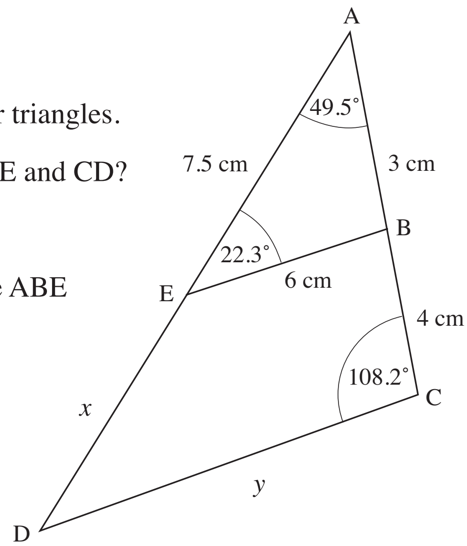
(ii) ABE to BCDE.

6. (a) Explain why ABE and ACD are similar triangles.

(b) What can be deduced about the lines BE and CD?

(c) Find the lengths x and y.

(d) Find the ratio of the area of the triangle ABE to the area of the quadrilateral BCDE.



7. A bottle has a height of 8 cm and a volume of 30 cm^3. Find the volume of similar bottles of heights:

   (a) 12 cm (b) 10 cm (c) 20 cm.

8. A box has a volume of 50 cm^3 and a width of 6 cm. A similar box has a width of 12 cm.

   (a) Find the volume of the larger box.

   (b) How many times bigger is the surface area of the larger box?

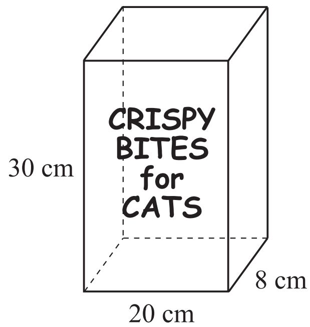
9. A packet has the dimensions shown in the diagram.

   All the dimensions are increased by 20%.

   (a) Find the percentage increase in:

      (i) surface area (ii) volume.

   (b) Find the percentage increase needed in the dimensions of the packet to increase the volume by 50%.



10. Two similar cans have volumes of 400 cm^3 and 1350 cm^3.

   (a) Find the ratio of the heights of the cans.

   (b) Find the ratio of the surface areas of the cans.

11. One box has a surface area of 96 cm^2 and a height of 4 cm. A second similar box has a volume of 1728 cm^3 and a surface area of 864 cm^2.

   Find:

   (a) the height of the larger box (b) the volume of the smaller box.

12.

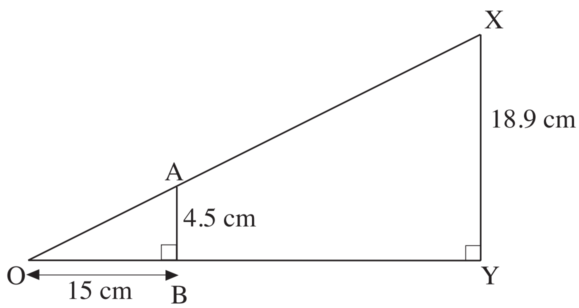
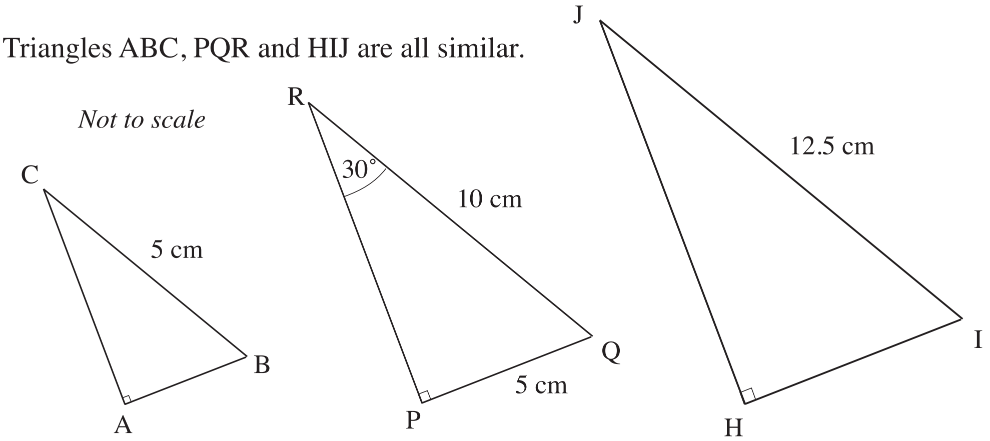


*Diagram **not** accurately drawn*

   (a) Calculate the length of OY.

   (b) Calculate the size of angle XOY.

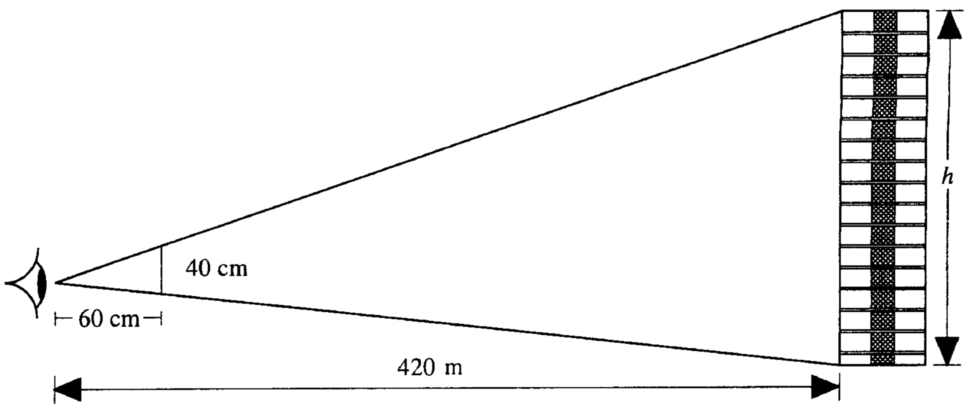
13. Triangles ABC, PQR and HIJ are all similar.



(a) Calculate the length of AB. (b) What is the size of angle B?

(c) Calculate the length of HJ.

14.

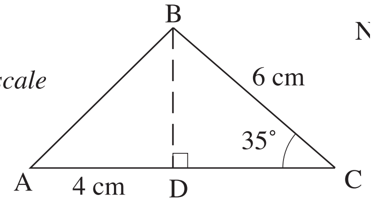


I stood 420 m away from the tallest building in Singapore. I held a piece of wood 40 cm long at arms length, 60 cm away from my eye. The piece of wood, held vertically, just blocked the building from my view.

Use similar triangles to calculate the height, h metres, of the building.

15. AD = 4 cm, BC = 6 cm, angle BCD = 35°.
BD is perpendicular to AC.

Not to scale



(a) Calculate BD.

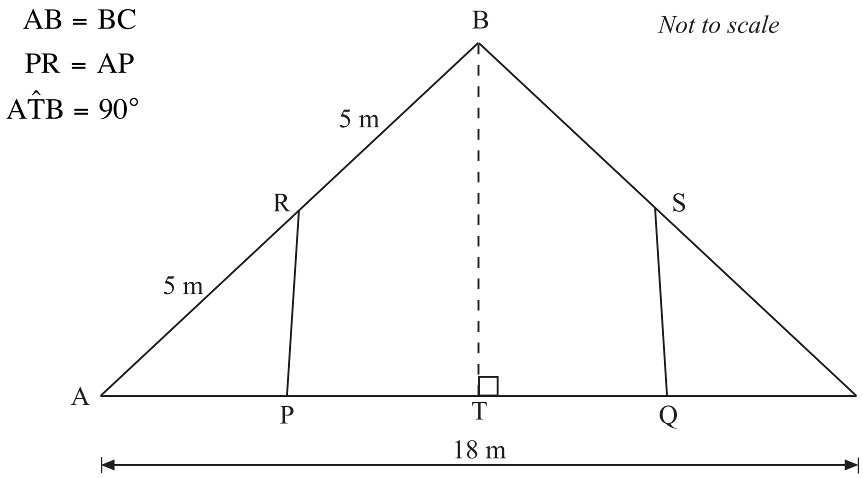
(b) Calculate angle BAC.

(c) Triangle $A'B'C'$ is similar to triangle ABC.
The area of triangle $A'B'C'$ is nine times the area of triangle ABC.

(i) What is the size of angle $A'B'C'$?

(ii) Work out the length of $B'C'$.

16. A roof has a symmetrical frame, with dimensions as shown.

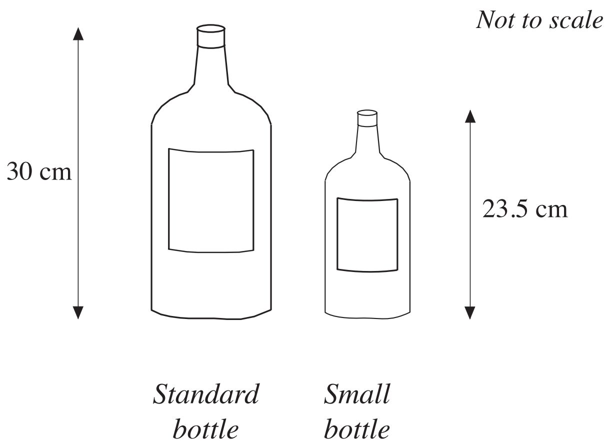


(a) (i) Write down a triangle which is similar to triangle ABC.

(ii) Calculate the length PR.

(b) Calculate the value of angle BAT.

17. Two wine bottles have similar shapes. The standard bottle has a height of 30 cm. The small bottle has a height of 23.5 cm.

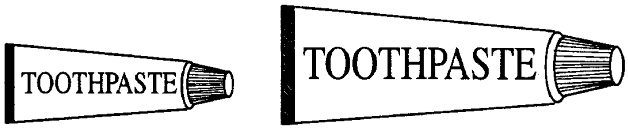


(a) Calculate the ratio of the areas of the bases of the two bottles. Give your answer in the form n : 1.

(b) What is the ratio of the volumes of the two bottles? Give your answer in the form n : 1.

(c) Is it a fair description to call the small bottle a 'half bottle'? Give a reason for your answer.

18. The normal size and selling price of small and medium toothpaste is shown.



SMALL SIZE 60 ml 50 cents

MEDIUM SIZE 135 ml \$1

A supermarket sells the toothpaste on special offer.

(a) The special offer small size has 20% more toothpaste for the same price. How much more toothpaste does it contain?

(b) The special offer medium size costs 90 cents for 135 ml. What is the special offer price as a fraction of the normal price?

(c) Calculate the number of ml per cent for each of these special offers. Which of these special offers gives better value for money?

You must show your working.

(d) (i) The 60 ml content of the small size has been given to the nearest 10 ml. What is the smallest number of ml it can contain?

(ii) The 135 ml content of the medium size has been given to the nearest 5 ml. What is the smallest number of ml it can contain?

19. (a) Two bottles of perfume are similar to each other. The heights of the bottles are 4 cm and 6 cm. The smaller bottle has a volume of 24 cm^3.

Calculate the volume of the larger bottle.

(b) Two bottles of aftershave are similar to each other. The areas of the bases of these bottles are 4.8 cm^2 and 10.8 cm^2. The height of the smaller bottle is 3 cm. Calculate the height of the larger bottle.