ANGLES, CIRCLES AND TANGENTS

Text

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Section

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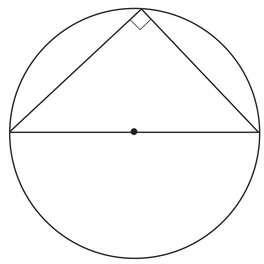
Angles, Circles and Tangents

1 Angles and Circles 1

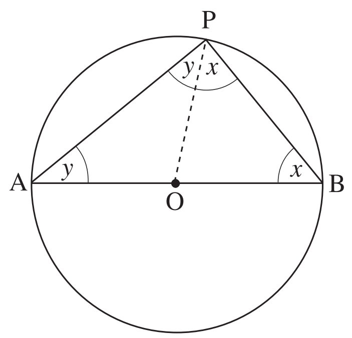


The following results are true in any circle.

When a triangle is drawn in a semi-circle as shown, the angle on the perimeter is always a right angle.

Proof

Join the centre, O, to the point, P, on the perimeter.



Since

OB = OP (equal radii)

then

angle OBP = angle OPB (= x, say)

Similarly, triangle AOP is also isosceles and

angle OAP = angle APO (= y, say).

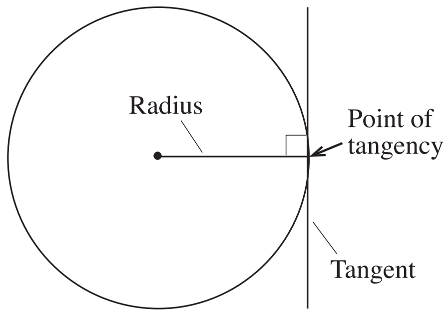
In triangle ABP, the sum of the angles must be $180°$.
Then

$$y + x + (x + y) = 180°$$

$$2x + 2y = 180° \quad \text{(collecting like terms)}$$

$$x + y = 90° \quad (\div 2)$$

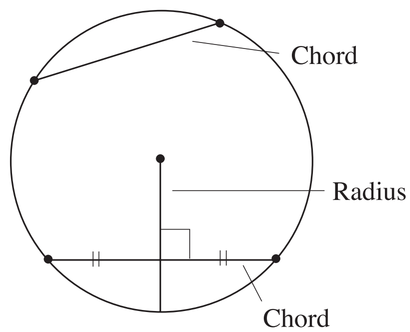
But angle APB = $x + y$, and this is a right angle.



A *tangent* is a line that touches only one point on the circumference of a circle.

This point is known as the *point of tangency.*

A tangent is always perpendicular to the radius of the circle.



A *chord* is a line joining any two points on the circle.

The *perpendicular bisector* is a second line that divides the first line in half and is at right angles to it.

The perpendicular bisector of a chord is always a radius of the circle.

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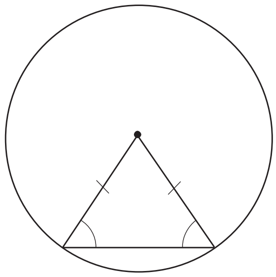
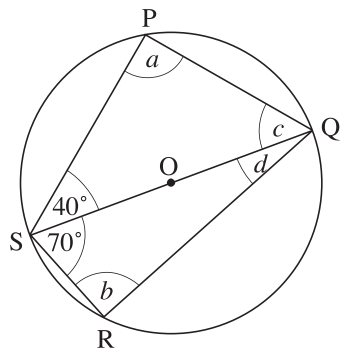


When the ends of a chord are joined to the centre of a circle, an isosceles triangle is formed, so the two angles marked are equal.

Worked Example 1

Find the angles marked with letters in the diagram, if O is the centre of the circle.



Solution

As both triangles are in semi-circles, angles a and b must each be $90°$.

The other angles can be found because the sum of the angles in each triangle is $180°$.

For the triangle PQS,

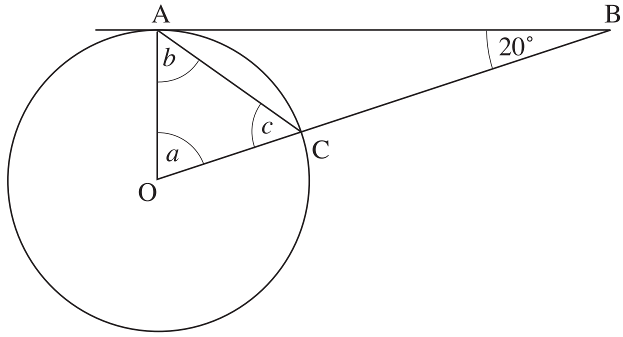
$$\begin{aligned} 40° + 90° + c &= 180° \\ c &= 180° - 130° \\ &= 50° \end{aligned}$$

For the triangle QRS,

$$\begin{aligned} 70° + 90° + d &= 180° \\ d &= 180° - 160° \\ &= 20° \end{aligned}$$

Worked Example 2

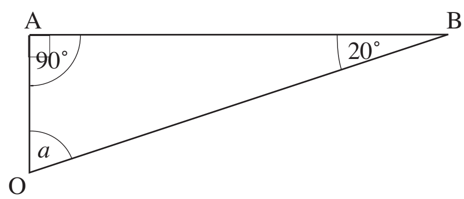
Find the angles a, b and c, if AB is a tangent and O is the centre of the circle.



Solution

First consider the triangle OAB. As OA is a radius and AB is a tangent, the angle between them is $90°$. So

$$\begin{aligned} 90° + 20° + a &= 180° \\ a &= 180° - 110° \\ &= 70° \end{aligned}$$



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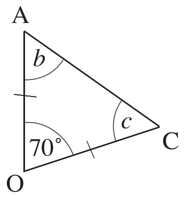
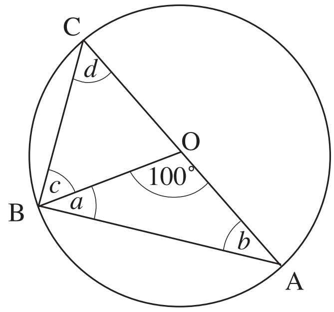
Then consider the triangle OAC. As OA and OC are both radii of the circle, it is an isosceles triangle with $b = c$.



So
$$2b + 70° = 180°$$
$$2b = 110°$$
$$b = 55°$$

and $c = 55°$.

Worked Example 3

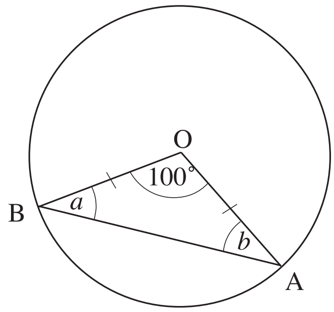


Find the angles marked in the diagram, where O is the centre of the circle.

Solution

First consider the triangle OAB.

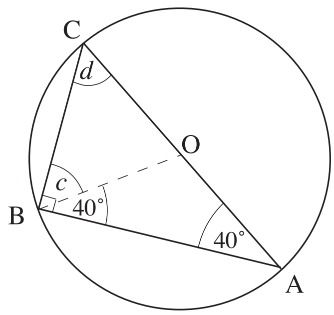
As the sides OA and OB are both radii, the triangle must be isosceles with $a = b$.



So $$a + b + 100° = 180°$$
but as $a = b$,
$$2a + 100° = 180°$$
$$2a = 80°$$
$$a = 40°$$
and $b = 40°$.

Now consider the triangle ABC.

As the line AC is a diameter of the circle, the angle ABC must be $90°$.
So
$$a + c = 90°$$
or
$$40° + c = 90°$$
$$c = 50°$$



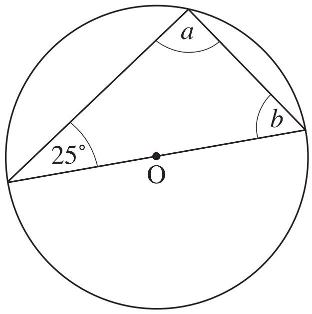
The angles in the triangle ABC must total $180°$, so

$$40° + 90° + d = 180°$$
$$d = 50°$$

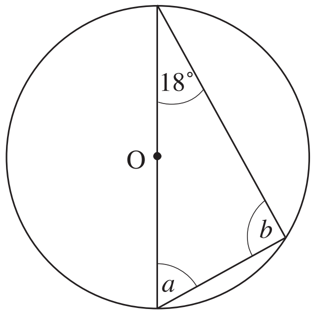
Exercises

1. Find the angles marked with a letter in each of the following diagrams. In each case the centre of the circle is marked O.

(a)



(b)



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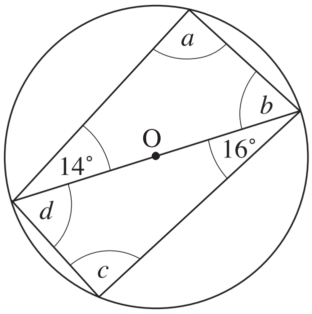
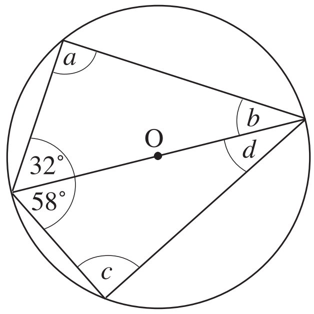
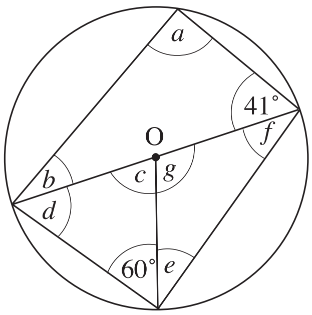


(d)



(e)



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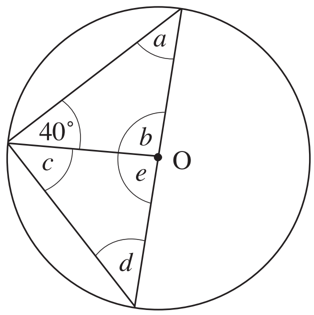
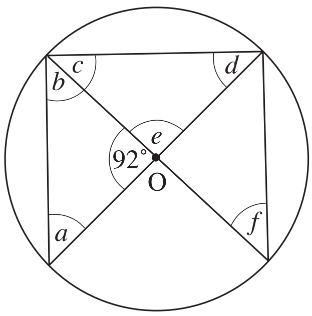
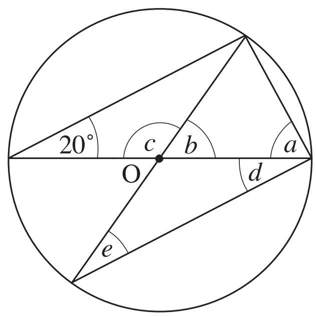


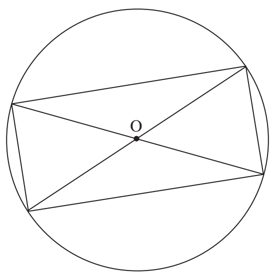
(g)



(h)



2. Copy the diagram opposite, and mark every right angle, if O is the centre of the circle.



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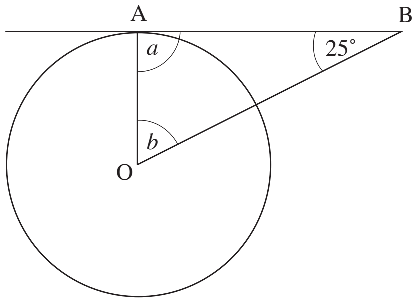
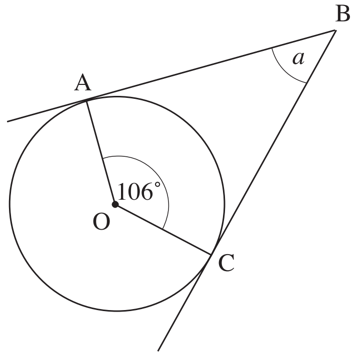
3. Find the angles marked with letters in each diagram below, if O is the centre of the circle.

(a)



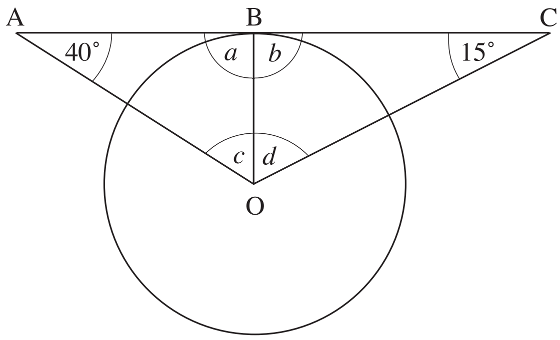
AB is a tangent

(b)



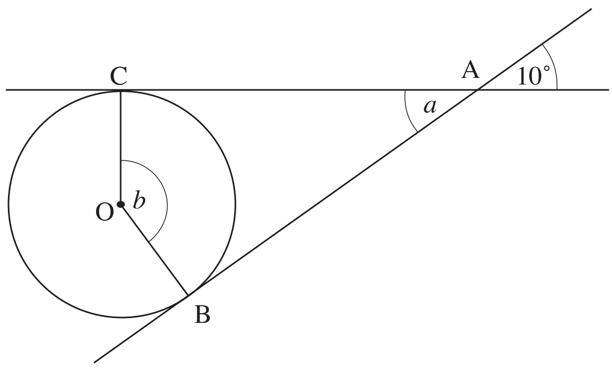
AB and BC are tangents

(c)



AC is a tangent

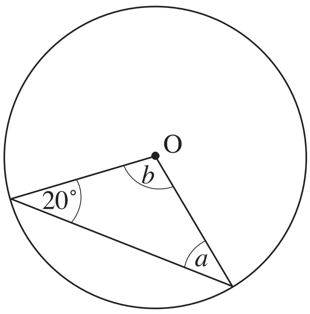
(d)



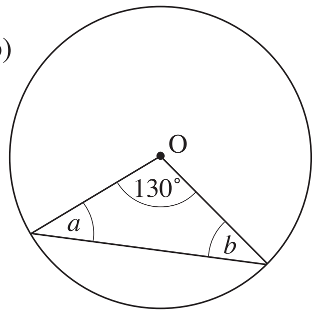
AB and AC are tangents

4. Find the angles marked with letters in each of the following diagrams, if O is the centre of the circle.

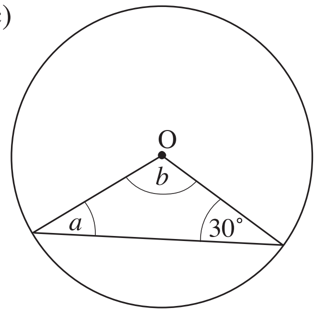
(a)



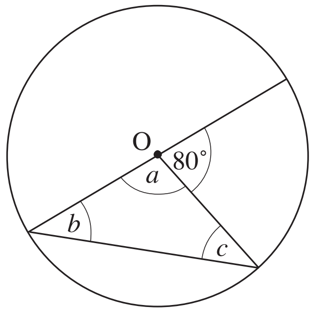
(b)



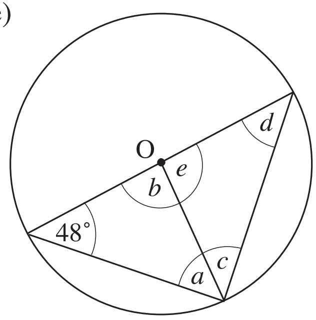
(c)



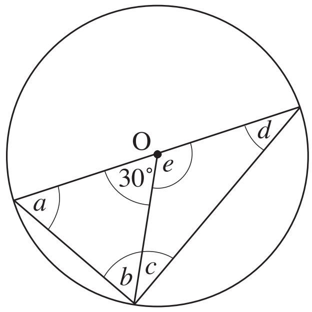
(d)



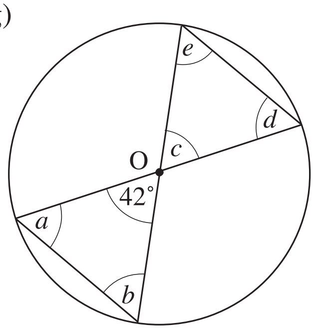
(e)



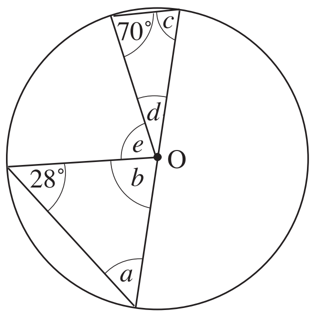
(f)



(g)

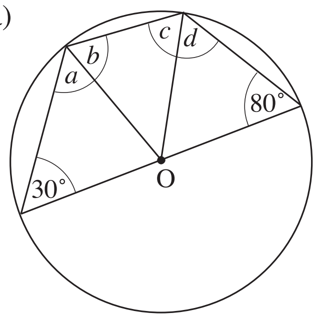


(h)

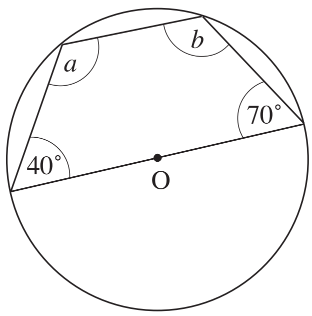


5. Find each of the marked angles if O is the centre of the circle.

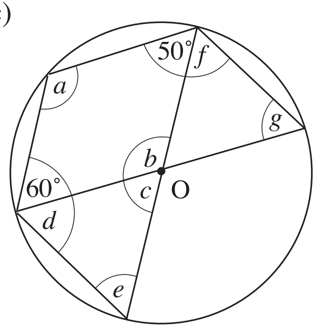
(a)



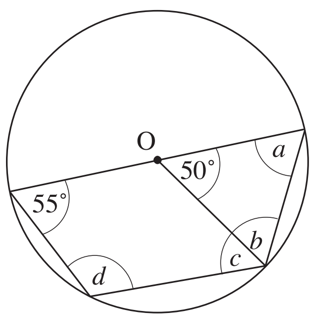
(b)



(c)

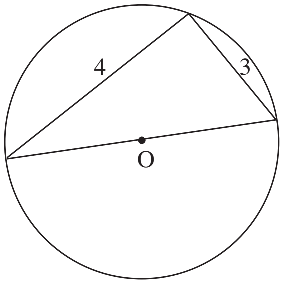


(d)

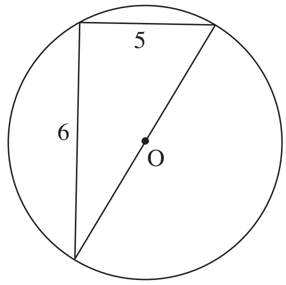


6. Find the diameter of each circle below, if O is the centre of the circle.

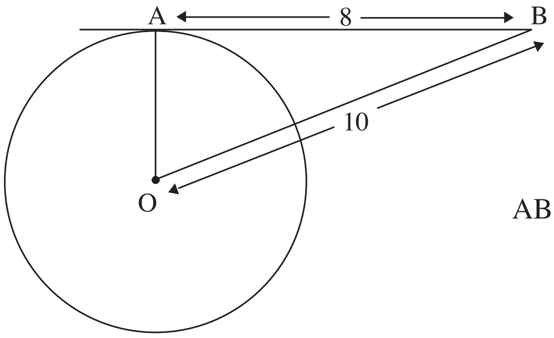
(a)



(b)

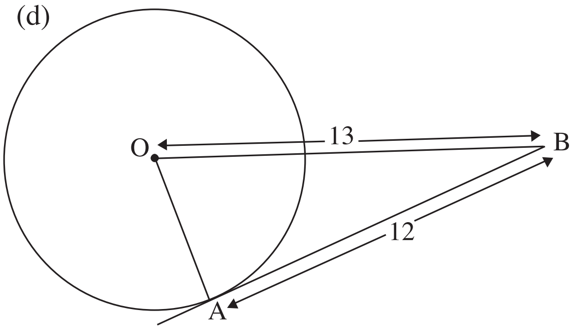


(c)



AB is a tangent

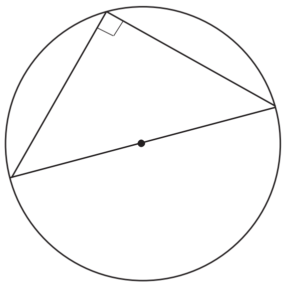
(d)



AB is a tangent

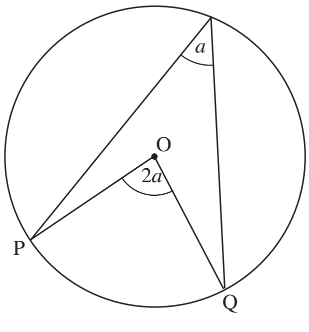
2 Angles and Circles 2

There are a number of important geometric results based on angles in circles. (The first you have met already.)



Any angle subtended at the circumference from a diameter is a right angle.

The angle subtended by an arc, PQ, at the centre is twice the angle subtended at the circumference.



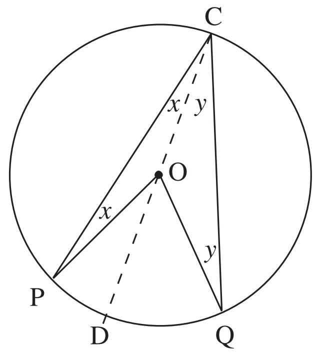
Proof

OP = OC (equal radii), so

$$\text{angle CPO} = \text{angle PCO} \quad (= x, \text{say}).$$

Similarly,

$$\text{angle CQO} = \text{angle QCO} \quad (= y, \text{say}).$$



Now, extending the line CO to D, say, note that

$$\text{angle POD} = x + x$$
$$= 2x$$

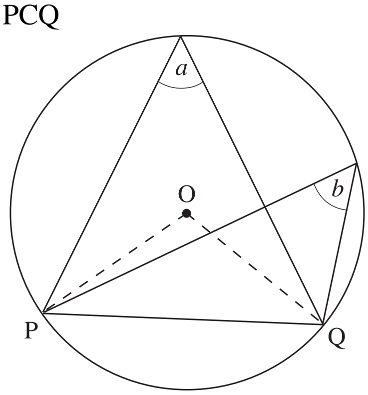
and, similarly,

$$\text{angle QOD} = y + y$$
$$= 2y$$

Hence,

$$\text{angle POQ} = 2x + 2y$$
$$= 2(x + y)$$
$$= 2 \times \text{angle PCQ}$$

as required.

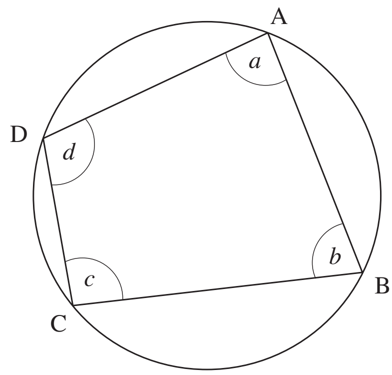


> Angles subtended at the circumference by a chord (on the same side of the chord) are equal;
> that is, in the diagram $a = b$.

Proof

The angle at the centre is $2a$ or $2b$ (according to the first result).

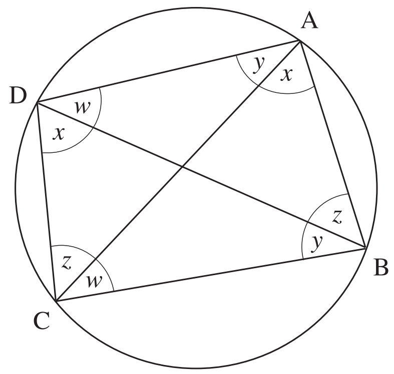
Thus $2a = 2b$ or $a = b$, as required.



> In *cyclic quadrilaterals* (quadrilaterals where all 4 vertices lie on a circle), opposite angles sum to 180° *; that is*
>
> $$a + c = 180°$$
> and $$b + d = 180°$$

Proof

Construct the diagonals AC and BD, as shown.



Then label the angles subtended by AB as w; that is

$$\text{angle ADB} = \text{angle ACB} \quad (= w)$$

Similarly for the other chords, the angles being marked x, y and z as shown.

Now, in triangle ABD, the sum of the angles is $180°$,
so

$$w + z + (x + y) = 180°$$

You can rearrange this as

$$(x + w) + (y + z) = 180°$$

which shows that

$$\text{angle CDA} + \text{angle CBA} = 180°$$

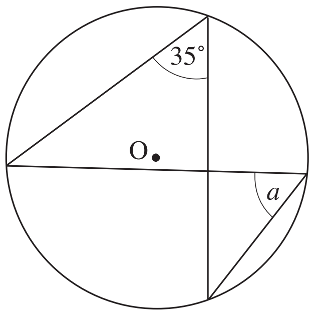
proving one of the results.

The other result follows in a similar way.

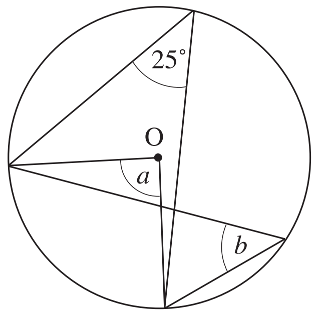
Worked Example 1

Find the angles marked in the diagrams. In each case O is the centre of the circle.

(a)



(b)



Solution

(a) As both angles are drawn on the same chord, the angles are equal, so

$$a = 35°$$

(b) Angle b and the $25°$ angle are drawn on the same chord, so

$$b = 25°$$

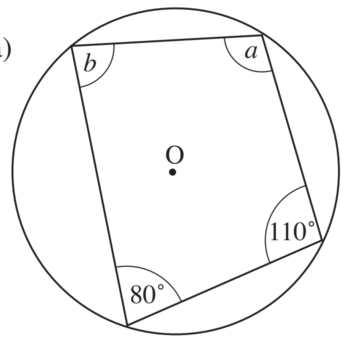
Angle a is drawn at the centre O on the same chord as the $25°$ angle, so

$$\begin{aligned} a &= 2 \times 25° \\ &= 50° \end{aligned}$$

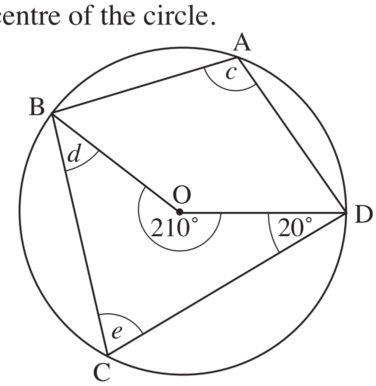
Worked Example 2

Find the angles marked in the diagrams. O is the centre of the circle.

(a)



(b)



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Solution

(a) Opposite angles in a cyclic quadrilateral add up to $180°$. So

$$a + 80° = 180°$$
$$a = 100°$$

and

$$b + 110° = 180°$$
$$b = 70°$$

(b) Consider the angles c and $210°$. Since the angle at the centre is double the angle in a segment drawn on the same arc,

$$2c = 210°$$
$$c = 105°$$

Angles c and e add up to $180°$ because they are opposite angles in a cyclic quadrilateral.

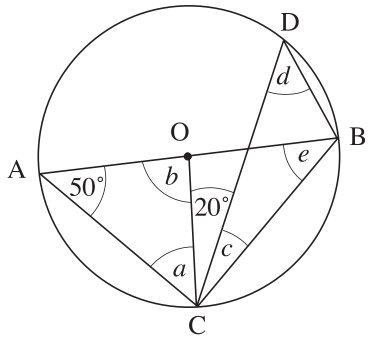
$$c + e = 180°$$
$$105° + e = 180°$$
$$e = 180° - 105°$$
$$= 75°$$

Consider the quadrilateral BODC. The four angles in any quadrilateral add up to $360°$. So

$$d + e + 210° + 20° = 360°$$
$$d = 360° - 210° - 20° - e$$
$$= 130° - c$$
$$= 130° - 75°$$
$$= 55°$$

Worked Example 3

In the diagram the line AB is a diameter and O is the centre of the circle. Find the angles marked.



Solution

Consider triangle OAC. Since OA and OC are radii, triangle OAC is isosceles. So

$$a = 50°$$

The angles in a triangle add to $180°$, so for triangle OAC,

$$a + b + 50° = 180°$$
$$b = 180° - 50° - a$$
$$= 80°$$

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Since AB is a diameter of the circle, the angle ACB is a right angle, so

$$a + 20° + c = 90°$$
$$c = 90° - 20° - a$$
$$= 20°$$

Angle d and angle OAC are angles in the same segment, so

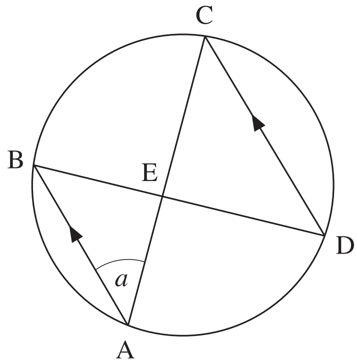
$$d = \text{angle OAC}$$
$$= 50°$$

Angle e is drawn on the same arc as the angle at the centre, AOC, so

$$b = 2e$$
$$e = \tfrac{1}{2}b$$
$$= 40°$$

Worked Example 4

In the diagram the chords AB and CD are parallel.
Prove that the triangles ABE and DEC are isosceles.



Solution

Angles a and BDC are angles in the same segment, so

$$\text{angle BDC} = a$$

Since AB and DC are parallel, angles a and ACD are equal alternate angles,

$$\text{angle ACD} = a = \text{angle BDC}$$

Hence in triangle DEC, the base angles at C and D are equal, so the triangle is isosceles.

The angle at B, angle ABD, equals the angle at C, angle ACD, because they are angles in the same segment:

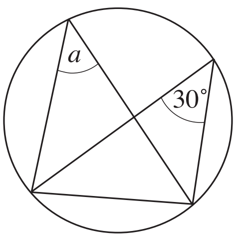
$$\text{angle ABD} = \text{angle ACD} = a$$

Hence triangle ABE is isosceles, since the angles at A and B are equal.

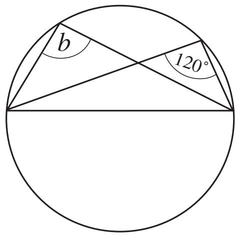
Exercises

1. Find all the angles marked with a letter in each of the following diagrams. In each case the centre of the circle is marked O. *Give reasons for your answers.*

(a)



(b)



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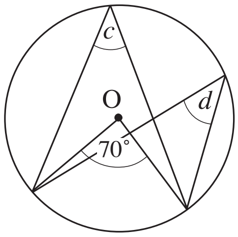
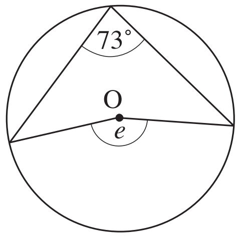
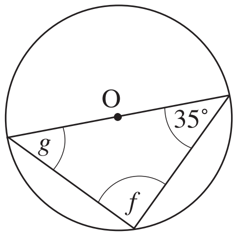
(c)



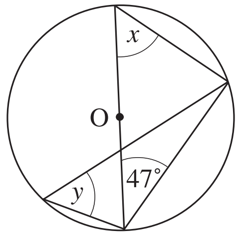
(d)



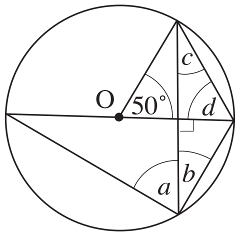
(e)



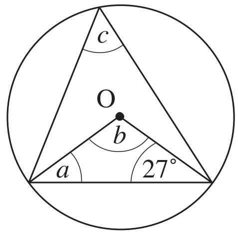
(f)



(g)



(h)

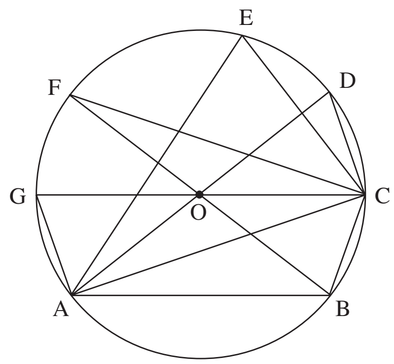


2. In the diagram, O is the centre of the circle and AOD, BOF and COG are diameters.

(a) Identify the equal angles.

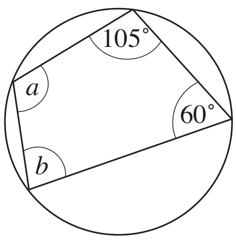
(b) Identify the right angles.

Give reasons for your answers.

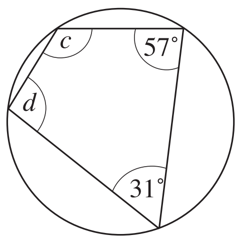


3. Find all of the angles marked with a letter in each of the following diagrams. *Give reasons for your answers.*

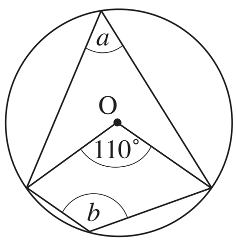
(a)



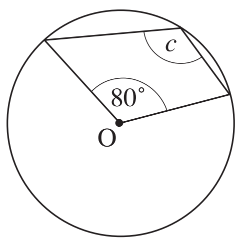
(b)



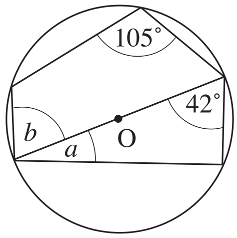
(c)



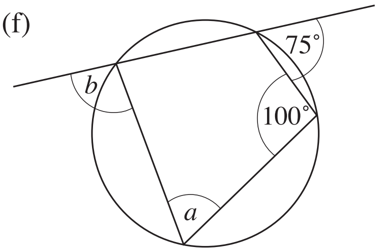
(d)



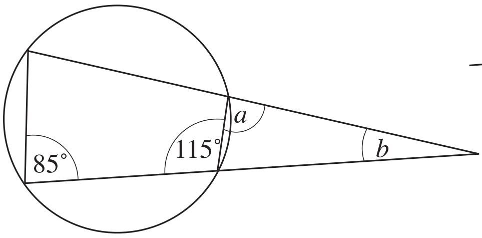
(e)



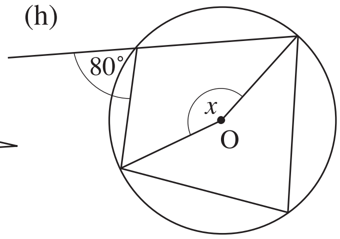
(f)



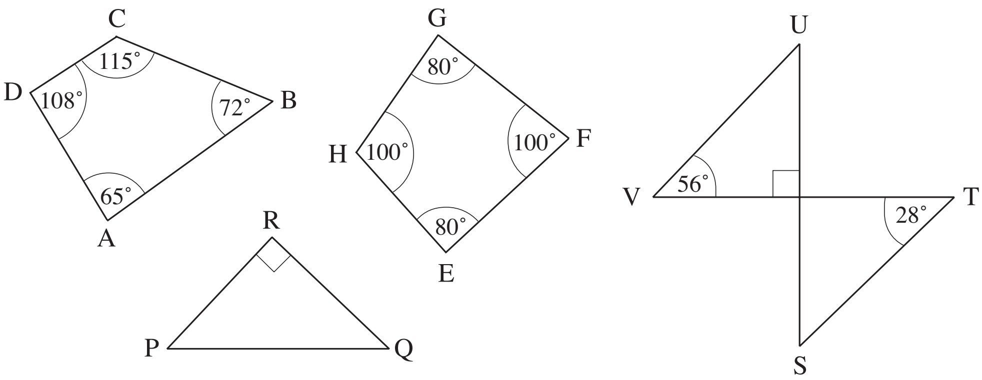
(g)



(h)

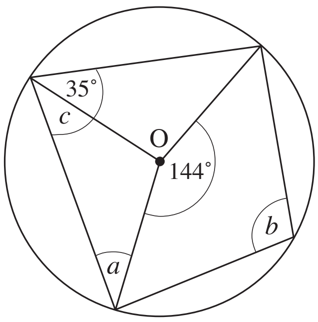


4. Which of the following points are concyclic points?

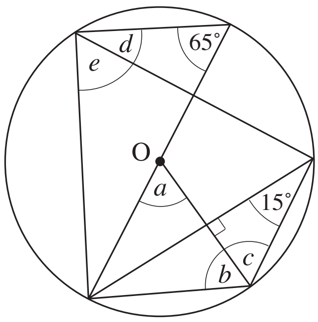


5. Find all the angles marked with a letter in the following diagrams. In each case, the point O is the centre of the circle.

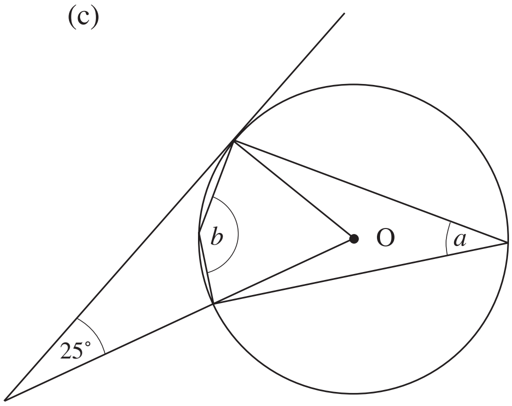
(a)



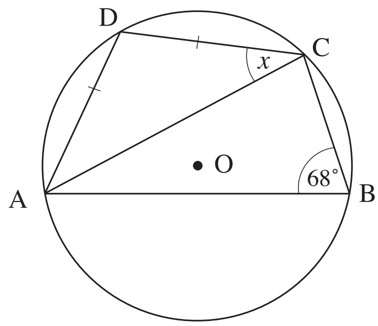
(b)



(c)



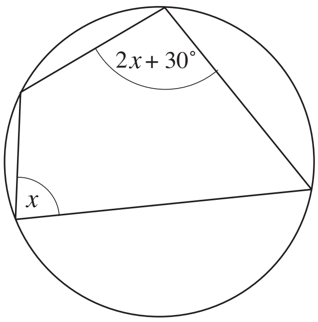
(d)



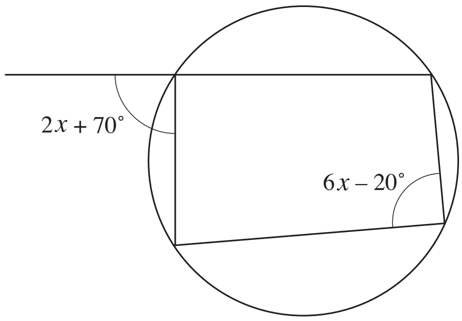
AD = DC

6. Find the value of x in each of the following diagrams.

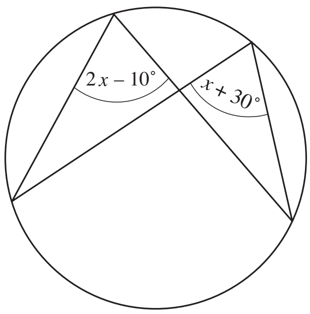
(a)



(b)

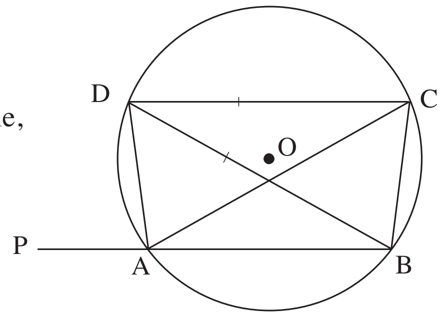


(c)

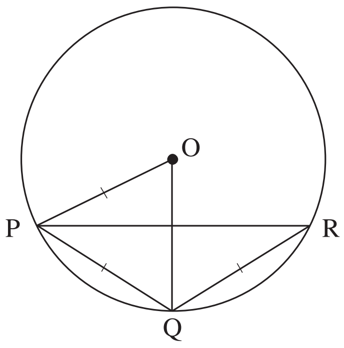


7. In the diagram, O is the centre of the circle, BD = DC and PAB is a straight line.

Prove that AD bisects the angle CAP.



8.



In the diagram O is the centre of the circle,

OP = PQ = QR

Prove that OP and QR are parallel lines.

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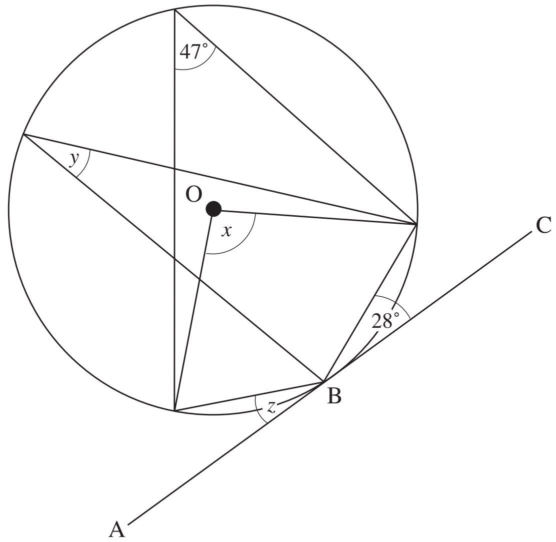
9. O is the centre of the circle.

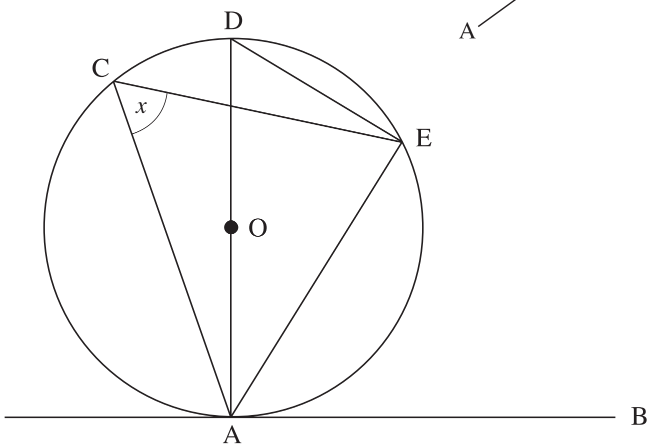
ABC is a tangent to the circle at B.

Not to scale



Work out the size of angles x, y and

10.



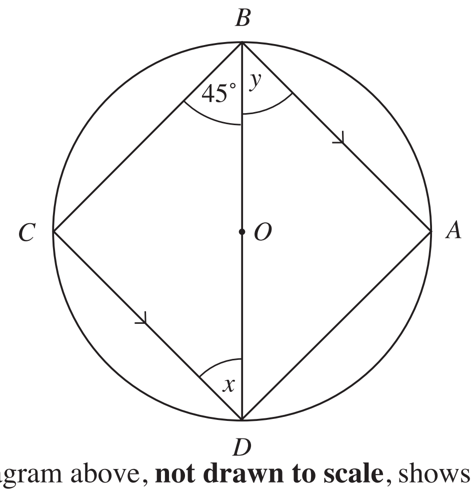
In the diagram, O is the centre of the circle, AD is a diameter and AB is a tangent.

Angle ACE = $x°$.

Find, in terms of x, the size of:

(a) angle ADE (b) angle DAE (c) angle EAB (d) angle AOE

11.



The diagram above, **not drawn to scale**, shows a circle, centre O. BA is parallel to CD and $C\hat{B}D = 45°$.

(a) Calculate, giving reasons, the values of x and y.

(b) Show that $ABCD$ is a square, giving the reasons for your answer.

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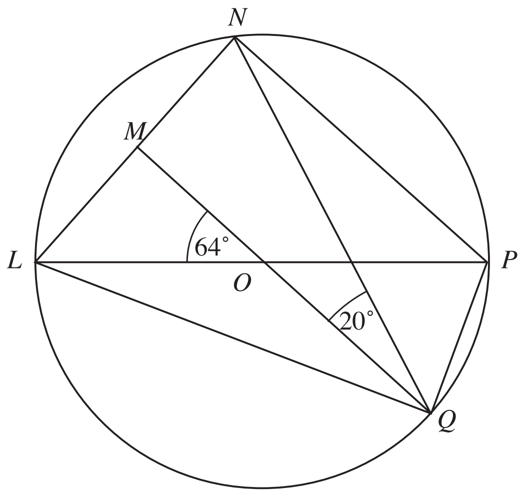
12.



The diagram shows a circle *LNPQ*, **not drawn to scale**, with centre *O*, angle $NQM = 20°$ and angle $MOL = 64°$.

Calculate, in degrees, giving reasons for your answers, the size of angles

(a) *OLQ*

(b) *NQP*

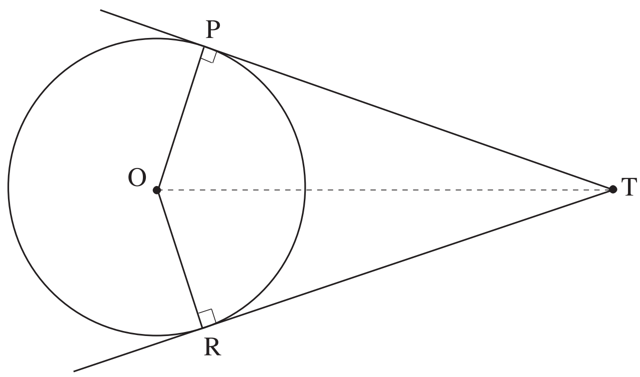
(c) *NLP*

(d) *NPL*

3 Circles and Tangents

Some important results are stated below.

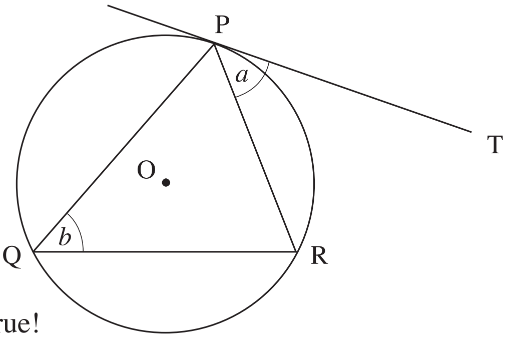
1. If two tangents are drawn from a point T to a circle with centre O, and P and R are the points of contact of the tangents with the circle, then, using symmetry,



(a) PT = RT

(b) Triangles TPO and TRO are congruent.

2. The angle between a tangent and a chord equals an angle at the circumference subtended by the same chord; e.g. $a = b$ in the diagram.



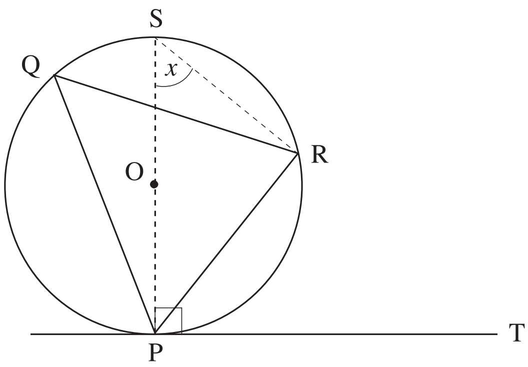
This is known as the

alternate segment theorem

and needs a proof, as it is not obviously true!

Proof

Construct the diameter POS, as shown.



We know that

$$\text{angle SRP} = 90°$$

since PS is a diameter.

Now

angle PSR = angle PQR = $x°$, say,

so

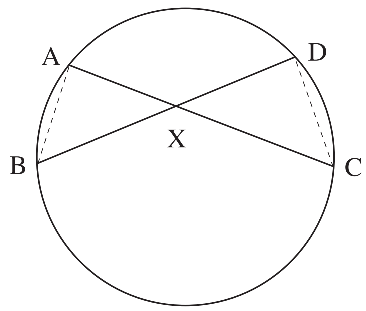
$$\text{angle SPR} = 180° - 90° - x$$
$$= 90° - x°$$

But

$$\text{angle RPT} = 90° - (\text{angle SPR})$$
$$= 90° - (90° - x°)$$
$$= x°$$
$$= \text{angle PQR}$$

and the result is proved.

3. For any two intersecting chords, as shown,



$$\boxed{AX \times CX = BX \times DX}$$

The proof is based on similar triangles.

Proof

In triangles AXB and DXC,

angle BAC = angle BDC (equal angles subtended by chord BC)

and

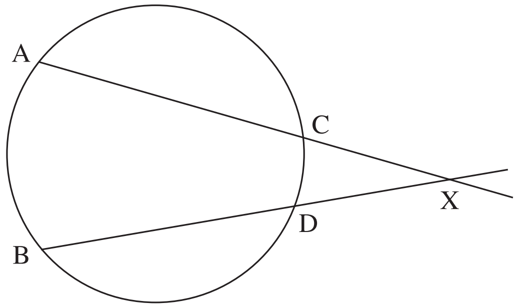
angle ABD = angle ACD (equal angles subtended by chord AD)

As AXB and DXC are similar,

$$\frac{AX}{BX} = \frac{DX}{CX} \quad \Rightarrow \quad AX.CX = BX.DX$$

as required.

This result will still be true even when the chords intersect *outside* the circle, as illustrated opposite.



How can this be proved?

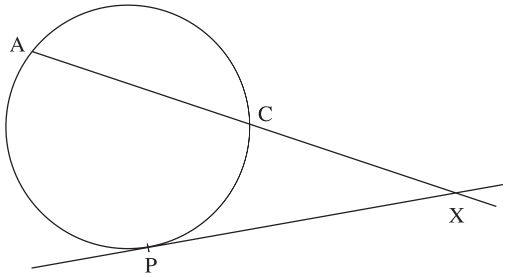
When the chord BD becomes a tangent,

and B and D coincide at the point P, then

$$AX \times CX = PX \times PX$$

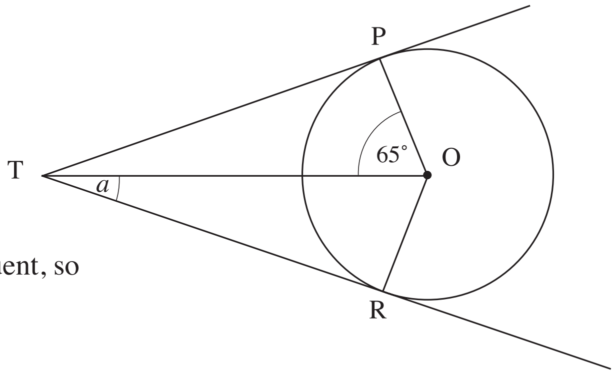
or

$$\boxed{AX \times CX = PX^2}$$



Worked Example 1

Find the angle a in the diagram.



Solution

The triangles TOR and TOP are congruent, so

$$\text{angle TOR} = 65°$$

Since TR is a tangent to the circle and OR is a radius,

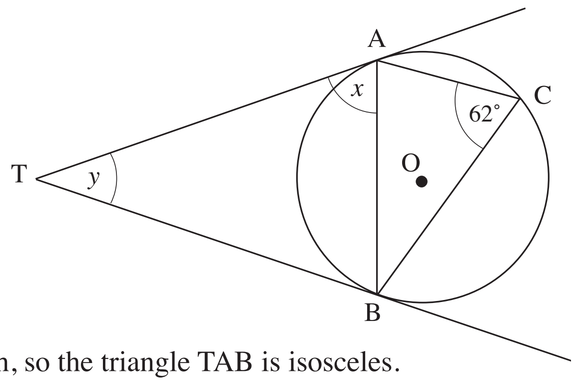
$$\text{angle TRO} = 90°.$$

Hence

$$a = 180° - 90° - 65°$$
$$= 25°$$

Worked Example 2

Find the angles x and y in the diagram.



Solution

The alternate angle segment theorem gives

$$x = 62°$$

The tangents TA and TB are equal in length, so the triangle TAB is isosceles. So

$$\text{angle ABT} = x = 62°$$

Hence

$$y + 62° + 62° = 180° \quad \text{(the angles in triangle TAB add up to } 180°\text{)}$$
$$y = 56°$$

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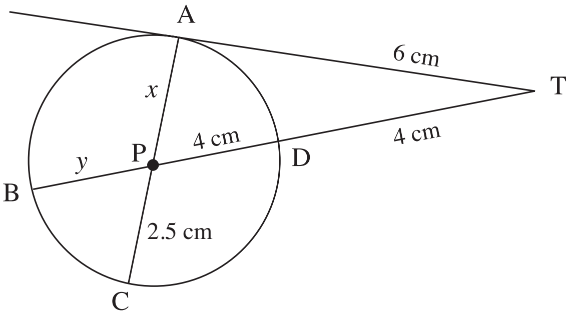
Worked Example 3

Find the unknown lengths in the diagram.



Solution

Since AT is a tangent,

$$AT^2 = BT.DT$$
$$36 = BT \times 4$$
$$BT = 9$$

Hence

$$y + 8 = BT = 9$$
$$y = 1 \text{ cm}$$

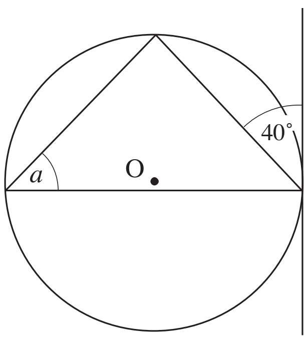
AC and BD are intersecting chords, so

$$AP.PC = BP.PD$$
$$2.5x = 1 \times 4$$
$$x = \frac{4}{2.5}$$
$$= 1.6 \text{ cm}$$

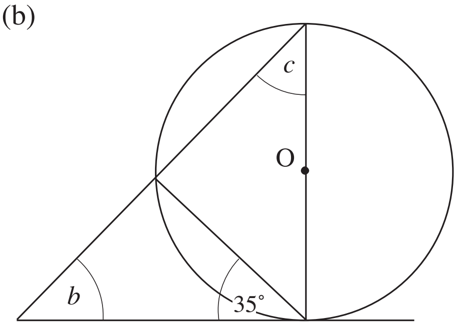
Exercises

1. Find the angles marked in the diagrams. In each case O is the centre of the circle.

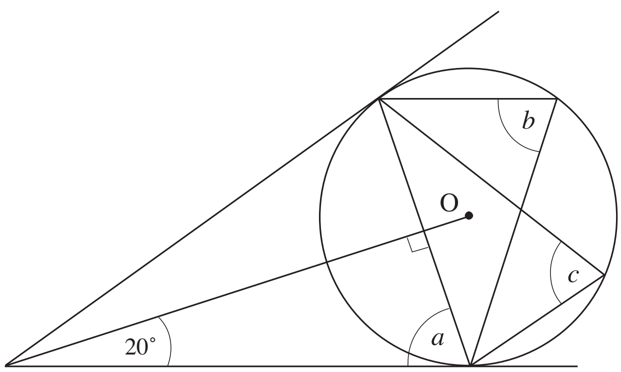
(a)



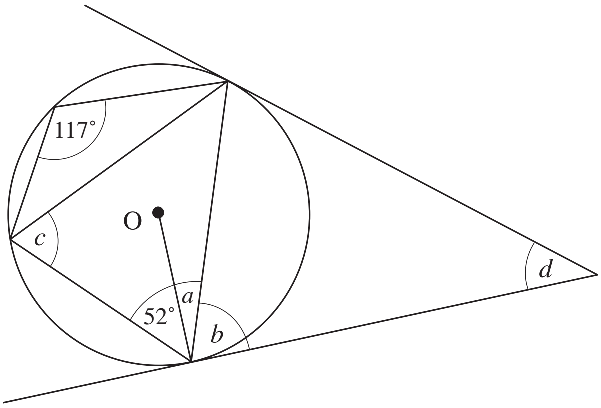
(b)



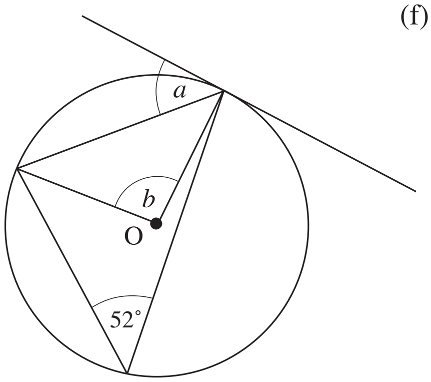
(c)



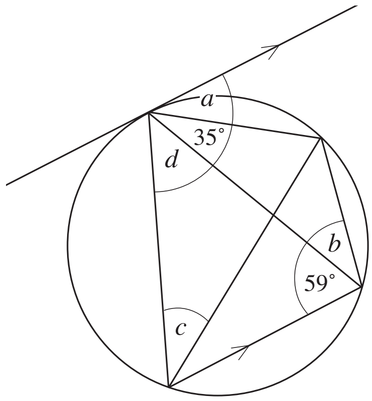
(d)



(e)

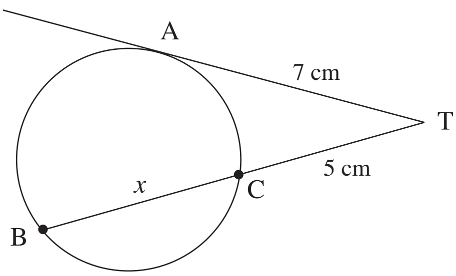


(f)

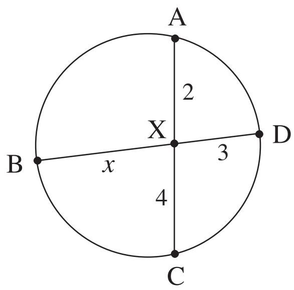


2. Find the unknown lengths in the following diagrams.

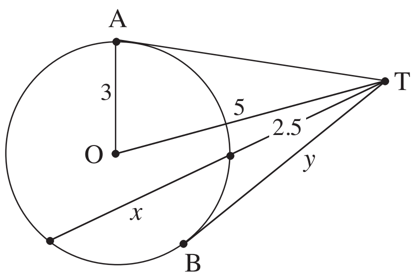
(a)



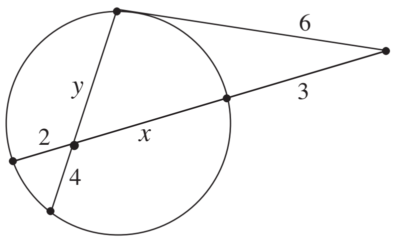
(b)



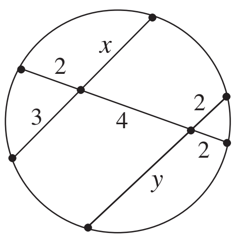
(c)



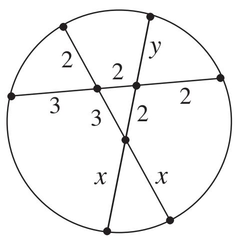
(d)



(e)



(f)



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3. In the diagram, TAD is a tangent to the circle.

   (a) Prove that $a = b$.

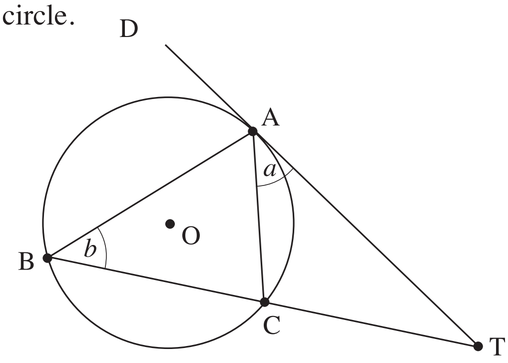
   (b) Show that triangles BTA and ACT are similar triangles.

   (c) If

   BC = 5 cm

   CT = 4 cm

   calculate the length of the tangent AT.

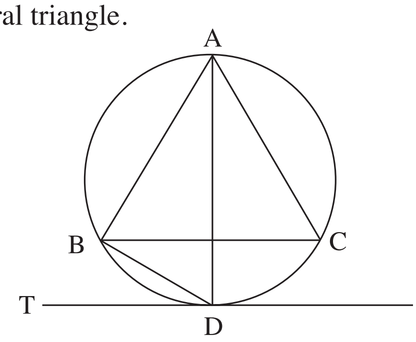


4. The triangle ABC in the diagram is an equilateral triangle.

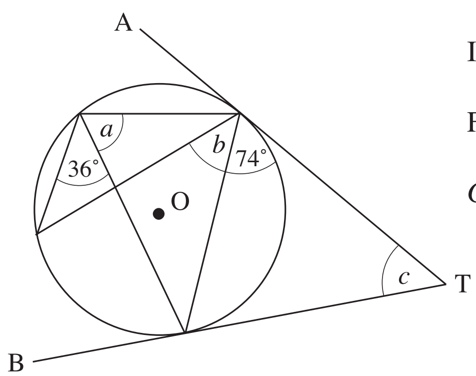
   The line AD bisects angle BAC.

   (a) Prove that the line AD is a diameter of the circle.

   (b) Hence find the angle BDT, where DT is a tangent to the circle.



5.



In the diagram, TA and TB are tangents.

Find the angles a, b and c.

Give reasons for your answers.

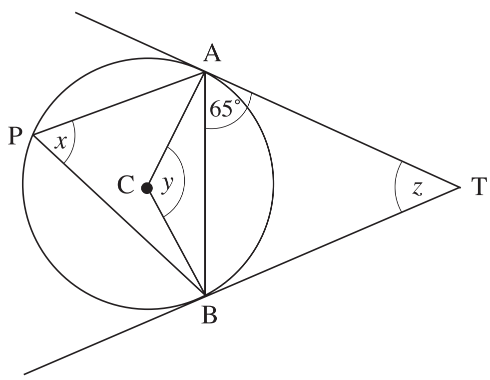
6. AT and BT are tangents to the circle, centre C.

   P is a point on the circumference, as shown.

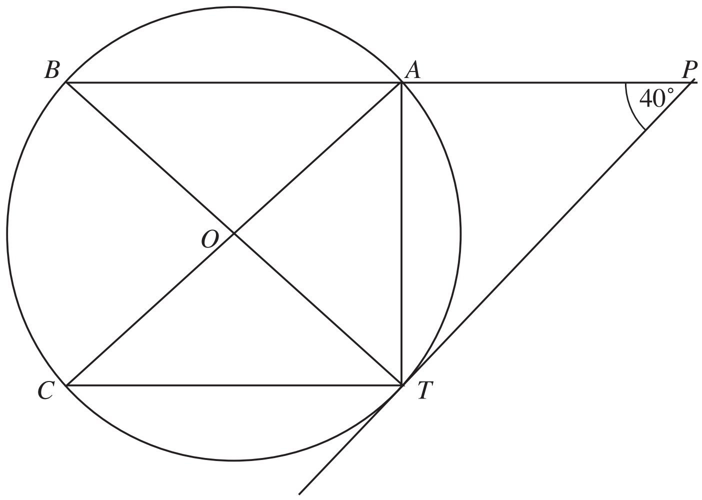
   Angle BAT = 65°

   Calculate the size of

   (a) x (b) y (c) z



7.



In the diagram above, **not drawn to scale**, *ABCT* is a circle. *AC* and *BT* are diameters. *TP,* the tangent at *T,* meets *BA* produced at *P,* so that $\angle APT = 40°$.

Calculate, **giving reasons for all statements**, the size of

(a) $\angle BTP$

(b) $\angle BAT$

(c) $\angle ABT$

(d) $\angle ACT$